1) $\underline{A} = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$ $\underline{B} = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$ $\underline{C} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \\ 0 & 1 \end{pmatrix}$

i) $2\underline{B} = \begin{pmatrix} 4 & -6 \\ 2 & 8 \end{pmatrix}$

$\underline{A} + \underline{C}$ is not possible

$\underline{C}\,\underline{A} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 4 \\ 1 & 2 \end{pmatrix}$

$\underline{A} - \underline{B} = \begin{pmatrix} 2 & 6 \\ 0 & -2 \end{pmatrix}$

ii) $\underline{C}\,\underline{A} = \begin{pmatrix} 3 & 1 \\ 2 & 4 \\ 1 & 2 \end{pmatrix}$

but $\underline{A}\,\underline{C}$ is not possible since $C$ would need the same number of rows as $A$ has columns $\therefore$ matrix multiplication is not commutative.

2) $z = a + bj$

i) $|z| = \sqrt{a^2 + b^2}$

$z^* = a - bj$

ii) $zz^* = (a+bj)(a-bj)$

$= a^2 + abj - abj + b^2$

$= a^2 + b^2$

$\therefore zz^* - |z|^2$

$= a^2 + b^2 - \left(\sqrt{a^2+b^2}\right)^2 = 0$

3) $\sum_{r=1}^{n} (r+1)(r-1)$

$= \sum_{r=1}^{n} (r^2 - 1)$

$= \frac{1}{6}n(n+1)(2n+1) - n$

$= \frac{1}{6}n(n+1)(2n+1) - \frac{6n}{6}$

$= \frac{1}{6}n\left[(n+1)(2n+1) - 6\right]$

$= \frac{1}{6}n\left[2n^2 + 3n + 1 - 6\right]$

$= \frac{1}{6}n(2n+5)(n-1)$

4) $\begin{pmatrix} 6 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

i) $6x - 2y = a$
$-3x + y = b$

ii) $\det\begin{pmatrix} 6 & -2 \\ -3 & 1 \end{pmatrix} = 6 - 6 = 0$

The eqns have no solutions unless $a = -2b$ in which case there are infinitely many solutions

5) $x^3 + 3x^2 - 7x + 1 = 0$

Roots $\alpha, \beta, \gamma$

i) $\alpha + \beta + \gamma = -3$
$\alpha\beta + \beta\gamma + \gamma\alpha = -7$
$\alpha\beta\gamma = -1$

5 ii) $2\alpha + 2\beta + 2\gamma = 2\sum\alpha = -6$

$2\alpha 2\beta + 2\beta 2\gamma + 2\alpha 2\gamma = 4\sum\alpha\beta$
$= -28$

$2\alpha 2\beta 2\gamma = 8\sum\alpha\beta\gamma = -8$

Eqn is

$$x^3 + 6x^2 - 28x + 8 = 0$$

6) Prove $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$

When $n = 1$ $\frac{1}{1 \times 2} = \frac{1}{2} = \frac{1}{1+1}$

$\therefore$ formula is true when $n = 1$

Assume it is true for some $n = k$

then $\sum_{r=1}^{k} \frac{1}{r(r+1)} = \frac{k}{k+1}$

But then $\sum_{r=1}^{k+1} \frac{1}{r(r+1)}$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{(k+1)}{(k+2)}$$

$$= \frac{(k+1)}{(k+1)+1}$$

which is the same formula with $k+1$ replacing $k$

So if formula is true for $n = k$, it is true for $n = k+1$

Since it is true for $n = 1$, by mathematical induction it is true for all $n$ where $n$ is a positive integer

7) $y = \frac{3+x^2}{4-x^2}$

i) $y = 0 \Rightarrow 3 + x^2 = 0$

$\Rightarrow x^2 = -3$

which has no real solution

ii) $y = \frac{3+x^2}{(2+x)(2-x)}$

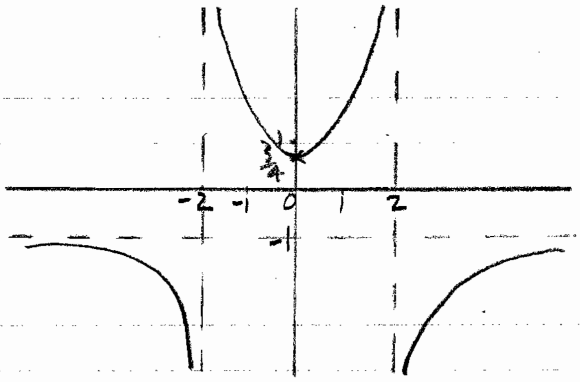
Asymptotes $x = -2$
$x = 2$
$y = -1$

iii) when $x = 100$, $y \approx -1.0007$

when $x = -100$, $y \approx -1.007$

For both +ve and −ve large $x$ values, $y$ approaches $-1$ from below

7 iv)




v)

$$\frac{3+x^2}{4-x^2} \le -2$$

Solve $3+x^2 = -2(4-x^2)$

$$3+x^2 = -8+2x^2$$

$$0 = x^2 - 11$$

$$11 = x^2$$

$$x = \pm\sqrt{11}$$

From graph solution is

$-\sqrt{11} < x < -2$ or $2 < x < \sqrt{11}$

8)

$\alpha = 1+j$ satisfies

$$z^3 + 3z^2 + pz + q = 0$$

i) $\alpha^2 = (1+j)(1+j)$

$$= 1 + 2j + j^2$$

$$\alpha^2 = +2j$$

$$\alpha^3 = 2j(1+j)$$

$$= 2j + 2j^2$$

$$\alpha^3 = -2+2j$$

$\therefore (-2+2j) + 3(2j) + p(1+j) + q = 0$

$$-2 + 2j + 6j + p + pj + q = 0$$

Equating Re and Im parts

$$-2 + p + q = 0 \qquad (1)$$

$$2 + 6 + p = 0 \qquad (2)$$

From (2) $p = -8$

Subst for $p$ in (1)

$$-2 - 8 + q = 0$$

$$\Rightarrow q = 10$$

ii)

$1+j$ a root $\Rightarrow 1-j$ a root

$$(z-(1+j))(z-(1-j))$$

$$= ((z-1)-j)((z-1)+j)$$

$$= (z-1)^2 + 1^2$$

$$= z^2 - 2z + 2$$

$$\begin{array}{r} z+5 \\ z^2-2z+2 \overline{\big)\, z^3 + 3z^2 - 8z + 10} \\ \underline{z^3 - 2z^2 + 2z} \\ 5z^2 - 10z + 10 \\ \underline{5z^2 - 10z + 10} \end{array}$$

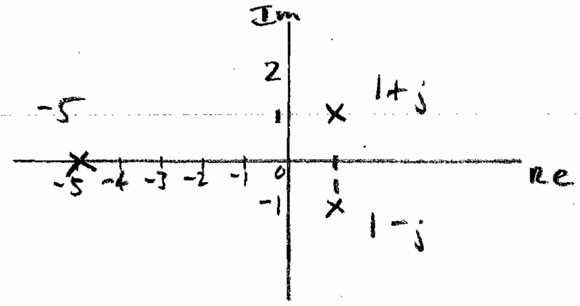
$$(z^2 - 2z + 2)(z+5) = 0$$

Roots $z = 1+j$

$z = 1-j$

$z = -5$

8(iii)




9)

i) Image of $(10, 50)$

$= (25, 50)$

ii) Image of $(x, y)$

$= \left(\frac{y}{2}, y\right)$

iii) Eqn of $\ell$ is $y = 6$

iv) Lines are $y = k$

where $k \in \mathbb{R}$

v) Lines are of form $y = 2x + c$

ie are parallel to $y = 2x$

vi)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{y}{2} \\ y \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{y}{2} \\ y \end{pmatrix}$$

Matrix $= \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$

vii) $\det \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$

$= 0 \times 1 - 0 \times \frac{1}{2} = 0$

Matrix is $\therefore$ singular

The matrix transforms every point in the $xy$-plane onto the line $y = 2x$

It is a many to one transformation