1) i) $\underline{A} = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$

$\underline{A}^{-1} = \frac{1}{\det \underline{A}} \begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix}$

$= \frac{1}{5} \begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix}$

$= \begin{pmatrix} \frac{2}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{4}{5} \end{pmatrix}$

1 ii)

$\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ -4 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 22 \\ -21 \end{pmatrix}$

$x = \frac{22}{5} \quad , \quad y = -\frac{21}{5}$

2)

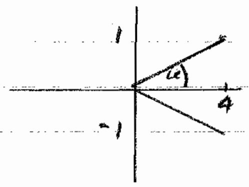
$x^2 - 8x + 17 = 0$

$x = \frac{8 \pm \sqrt{8^2 - 4 \times 17}}{2}$

$x = \frac{8 \pm \sqrt{-4}}{2}$

$x = \frac{8 \pm 2j}{2}$

$x = 4 + j \quad , \quad x = 4 - j$



$\alpha = \tan^{-1}\left(\frac{1}{4}\right) = 0.245$ radians

$|4 + j| = |4 - j| = \sqrt{17}$

Roots are:

$x = \sqrt{17}(\cos 0.245 + j \sin 0.245)$

$x = \sqrt{17}(\cos 0.245 - j \sin 0.245)$

3) $\underline{M} = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$

$\begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$3x - y = x$

$2x = y$

Line of invariant points is

$y = 2x$

4) $x^2 - 2x + 4 = 0$

Roots α, β

i) $\alpha + \beta = 2 \quad , \quad \alpha\beta = 4$

ii) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= 2^2 - 8 = -4$

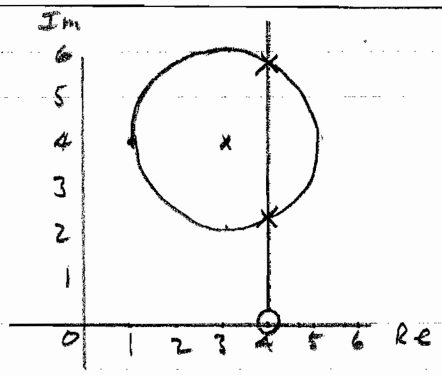
iii) Find eqn with roots $2\alpha, 2\beta$

$2\alpha + 2\beta = 2(\alpha+\beta) = 2\times 2 = 4$

$2\alpha \times 2\beta = 4\alpha\beta = 4\times 4 = 16$

Eqn is $x^2 - 4x + 16 = 0$

5) i



Circle centre $3+4j$ radius 2.

ii) Show $\arg(z-4) = \frac{\pi}{2}$

Half line from $4+0j$ excluding point $4+0j$

iii) Points marked by x s

6) Prove $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

When $n=1$ $r^3 = 1$

$\frac{1}{4}n^2(n+1)^2 = \frac{1}{4}1^2(2)^2 = 1$

$\therefore$ formula true when $n=1$

Assume true for some value $n=k$

Then $\sum_{r=1}^{k} = \frac{1}{4}k^2(k+1)^2$

So $\sum_{r=1}^{k+1} = \frac{1}{4}k^2(k+1)^2 + (k+1)^3$

$= (k+1)^2\left[\frac{1}{4}k^2 + k + 1\right]$

$= \frac{1}{4}(k+1)^2\left[k^2+4k+4\right]$

$= \frac{1}{4}(k+1)^2(k+2)^2$

$= \frac{1}{4}(k+1)^2((k+1)+1)^2$

which is the same formula with $k+1$ replacing k

$\therefore$ if formula is true for $n=k$, it is true for $n=k+1$

Since it is true for $n=1$ by mathematical induction it is true for all n where n is a positive integer

7) $\sum_{r=1}^{n} 3r(r-1)$

$= \sum_{r=1}^{n}(3r^2 - 3r)$

$= 3\sum_{r=1}^{n} r^2 - 3\sum_{r=1}^{n} r$

$= 3\times\frac{1}{6}n(n+1)(2n+1) - 3\times\frac{1}{2}n(n+1)$

$= \frac{1}{2}n(n+1)(2n+1) - \frac{3}{2}n(n+1)$

$= \frac{1}{2}n(n+1)\left[2n+1-3\right]$

$= \frac{1}{2}n(n+1)(2n-2)$

$= n(n+1)(n-1)$

Section B

8) $$y = \frac{x^2-4}{(3x-2)^2}$$

$$y = \frac{(x+2)(x-2)}{(3x-2)^2}$$

i) Asymptotes

$$x = \frac{2}{3}$$

and $$y = \frac{1}{9}$$

ii)

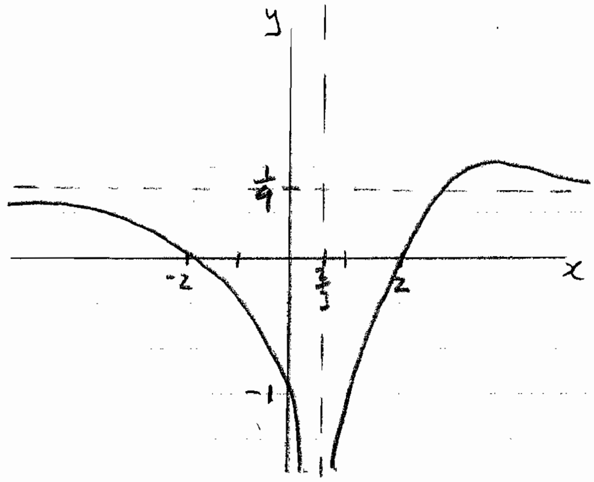
Asymptote $y = \frac{1}{9} = 0.11111$

when $x = 100$ $y \approx 0.1126$

when $x = -100$ $y \approx 0.1096$

For +ve large x $y \to \frac{1}{9}$ from above

For −ve large x $y \to \frac{1}{9}$ from below



iv)

$$\frac{x^2-4}{(3x-2)^2} \geqslant -1$$

$$\Rightarrow x^2 - 4 \geqslant -(3x-2)^2$$

$$x^2 - 4 \geqslant -\left[9x^2 - 12x + 4\right]$$

$$9x^2 - 12x + 4 + x^2 - 4 \geqslant 0$$

$$10x^2 - 12x \geqslant 0$$

$$2x(5x-6) \geqslant 0$$

Either $x \geqslant$ and $5x - 6 \geqslant 0$

$$\Rightarrow x \geqslant \frac{6}{5}$$

or $x \leq 0$ and $5x - 6 \leq 0$

$$\Rightarrow x \leq 0$$

Solution

$$x \leq 0 \quad \text{or} \quad x \geqslant \frac{6}{5}$$

9) $$x^4 + Ax^3 + Bx^2 + Cx + D = 0$$

has roots $2+j$ and $-2j$

i) Other roots $2-j$

and $2j$

ii) $$\Sigma\alpha = 2+j+2-j+2j-2j$$

$$\Sigma\alpha = 4$$

$$\begin{aligned}\Sigma\alpha\beta = &(2+j)(2-j)\\ &+(2+j)(2j)\\ &+(2+j)(-2j)\\ &+(2-j)(2j)\\ &+(2-j)(-2j)\\ &+(2j)(-2j)\end{aligned}$$

9 ii) cont

$= 4 + 1 + \cancel{4j} - 2 - \cancel{4j} + 2$
$+ \cancel{4j} + 2 - \cancel{4j} - 2 + 4$

$\Sigma\alpha\beta = 9$

$\Sigma\alpha\beta\gamma = \cancel{(2+j)(2-j)2j}$
$+ \cancel{(2+j)(2-j)(-2j)}$
$+ (2j)(-2j)(2+j)$
$+ (2j)(-2j)(2-j)$

$= 4(2+j) + 4(2-j)$

$= 8 + 4j + 8 - 4j = 16$

$\Sigma\alpha\beta\gamma = 16$

$\Sigma\alpha\beta\gamma\delta = (2+j)(2-j)(2j)(-2j)$
$= (4+1)(4) = 20$

$\Sigma\alpha = 4 \Rightarrow A = -4$

$\Sigma\alpha\beta = 9 \Rightarrow B = 9$

$\Sigma\alpha\beta\gamma = 16 \Rightarrow C = -16$

$\Sigma\alpha\beta\gamma\delta = 20 \Rightarrow D = 20$

9ii) Alternative Solution

If $2+j,\ 2-j,\ 2j,\ -2j$

are roots of $f(x) = 0$

$f(x) =$

$(x-(2+j))(x-(2-j))(x-2j)(x-(-2j))$

$= ((x-2)-j)((x-2)+j)(x-2j)(x+2j)$

$= ((x-2)^2+1)(x^2+4)$

$= (x^2-4x+5)(x^2+4)$

$= x^4 - 4x^3 + 5x^2 + 4x^2 - 16x + 20$

$= x^4 - 4x^3 + 9x^2 - 16x + 20$

$\Rightarrow A = -4$
$B = +9$
$C = -16$
$D = +20$

10)

i) $\dfrac{2}{r(r+1)(r+2)} = \dfrac{1}{r} - \dfrac{2}{r+1} + \dfrac{1}{r+2}$

$\displaystyle\sum_{r=1}^{n} \frac{2}{r(r+1)(r+2)}$

$= \frac{1}{1} + \frac{1}{2} + \cancel{\frac{1}{3}} + \cancel{\frac{1}{4}} + \cdots + \cancel{\frac{1}{n-1}} + \cancel{\frac{1}{n}}$

$- \frac{2}{2} - \cancel{\frac{2}{3}} - \cancel{\frac{2}{4}} - \cdots - \cancel{\frac{2}{n}} - \frac{2}{n+1}$

$+ \cancel{\frac{1}{3}} + \cancel{\frac{1}{4}} + \frac{1}{5} + \cdots + \cancel{\frac{1}{n}} + \frac{1}{n+1} + \frac{1}{n+2}$

$= 1 - \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}$

$= \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}$

$= \frac{1}{2} - \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$

$= \frac{1}{2} - \frac{n+2-(n+1)}{(n+1)(n+2)} = \frac{1}{2} - \frac{1}{(n+1)(n+2)}$

iii) $\displaystyle\sum_{r=1}^{\infty} \frac{2}{r(r+1)(r+2)} = \frac{1}{2}$

since $\frac{1}{n+1}$ and $\frac{1}{n+2} \to 0$ as $n \to \infty$

$\therefore \displaystyle\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)} = \frac{1}{4}$