MEI FP1 JUNE 2006

1) i) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$

Reflection in x-axis

ii) Rotation by $\theta°$ anticlockwise about origin

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Rotation by $90°$ anticlockwise about origin

$$\begin{pmatrix} \cos 90 & -\sin 90 \\ \sin 90 & \cos 90 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

iii) Rotation followed by reflection

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

(Note that first transformation required is on the right)

2) $2x^3 - 3x^2 + x - 2$

$\equiv (x+2)(Ax^2 + Bx + C) + D$

$\equiv Ax^3 + 2Ax^2$
$\quad + Bx^2 + 2Bx$
$\quad + Cx + 2C$
$\quad + D$

Equate coeff of x^3

$$2 = A$$

Equate coeff of x^2

$$-3 = 2A + B = 4 + B$$
$$-3 - 4 = B$$
$$-7 = B$$

Equate coeff of x

$$+1 = 2B + C = -14 + C$$
$$1 + 14 = C$$
$$15 = C$$

Equate coeff of constant

$$-2 = 2C + D = 30 + D$$
$$-2 - 30 = D$$
$$-32 = D$$

$\therefore$ $A = 2$
$B = -7$
$C = 15$
$D = -32$

MEI FP1 JUNE 2006

2) Alternative method algebraic long division

$$
\begin{array}{r}
2x^2 - 7x + 15 \\
x+2 \;\big)\; 2x^3 - 3x^2 + x - 2 \\
\underline{2x^3 + 4x^2} \quad\quad\quad \\
-7x^2 + x \quad\; \\
\underline{-7x^2 - 14x} \quad\; \\
+15x - 2 \\
\underline{+15x + 30} \\
-32
\end{array}
$$

$A = 2,\ B = -7,\ C = 15$

the remainder is $D = -32$

3) i) $z^3 + 4z^2 - 3z + 1 = 0$

$\alpha + \beta + \gamma = -4$

$\alpha\beta + \beta\gamma + \alpha\gamma = -3$

$\alpha\beta\gamma = -1$

ii) We require $\alpha^2 + \beta^2 + \gamma^2$

$(\alpha + \beta + \gamma)^2$

$= (\alpha + \beta + \gamma)(\alpha + \beta + \gamma)$

$$
\begin{aligned}
= \alpha^2 &+ \alpha\beta + \alpha\gamma \\
+ \beta^2 &+ \alpha\beta \quad\quad\quad + \beta\gamma \\
+ \gamma^2 & \quad\quad\quad + \alpha\gamma + \beta\gamma
\end{aligned}
$$

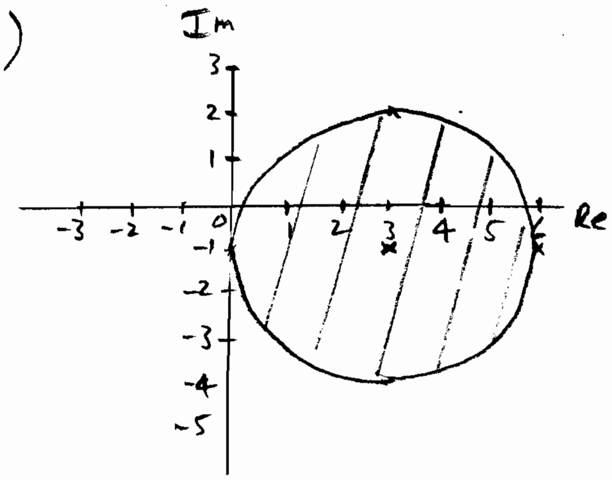
$= \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$\therefore (-4)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(-3)$

$16 = \alpha^2 + \beta^2 + \gamma^2 - 6$

$\alpha^2 + \beta^2 + \gamma^2 = 16 + 6 = 22$

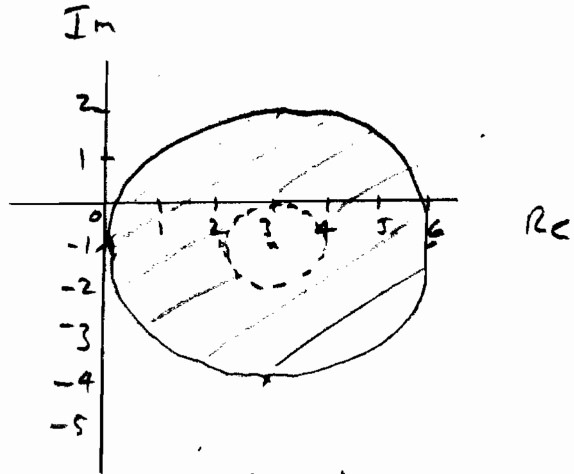
4) i)



Circular disc radius 3 centred on $(3 - j)$

$|z - (3 - j)| \le 3$

ii)



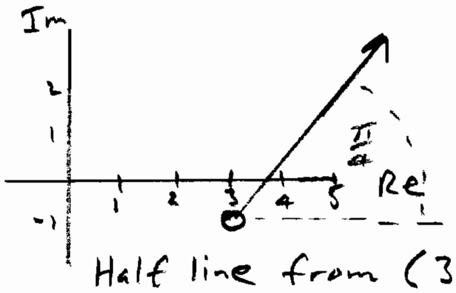
Annulus centre $(3 - j)$

Outer solid radius 3

Inner dotted radius 1

$1 < |z - (3 - j)| \le 3$

iii)



Half line from $(3 - j)$

MEI FP1 JUNE 2006

4iii) cont) at $\frac{\pi}{4}$ to real axis

$$\arg(3-j) = \frac{\pi}{4}$$

5)i) $$\underline{S} = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}$$

A) $$\begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1+2 \\ -3+4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\therefore \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is invariant

B) $$\det \underline{S} = -4 - -6 = 2$$

$$\underline{S}^{-1} = \frac{1}{2}\begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 \\ 3/2 & -\frac{1}{2} \end{pmatrix}$$

C) $$\begin{pmatrix} 2 & -1 \\ 3/2 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2-1 \\ 3/2-\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\therefore \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is invariant

5ii) If $\underline{T}$ is non-singular it has an inverse $\underline{T}^{-1}$

If $\begin{pmatrix} x \\ y \end{pmatrix}$ is invariant under $\underline{T}$

$$\underline{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \underline{T}^{-1}\underline{T}\begin{pmatrix} x \\ y \end{pmatrix} = \underline{T}^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \underline{I}_2\begin{pmatrix} x \\ y \end{pmatrix} = \underline{T}^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \underline{T}^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$$

so $\begin{pmatrix} x \\ y \end{pmatrix}$ is also invariant under $\underline{T}^{-1}$

6) Prove by induction

$$3 + 6 + 9 + \ldots 3\times 2^{n-1} = 3(2^n - 1)$$ for all positive integers n

When $n = 1$

$$3 = 3(2^1 - 1) = 3 \quad \checkmark$$

$\therefore$ true for $n = 1$

Assume true for $n = k$

$$3 + 6 + 9 + \ldots + 3\times 2^{k-1} = 3(2^k - 1)$$

Now consider $n = k+1$

$$3 + 6 + 9 + \ldots + 3\times 2^{k-1} + 3\times 2^{k+1-1}$$

$$= 3(2^k - 1) + 3\times 2^k$$

$$= 3\times 2^k - 3 + 3\times 2^k$$

$$= 3\times(2^k + 2^k) - 3$$

$$= 3\times 2(2^k) - 3$$

6 cont) $= 3 \times 2^{k+1} - 3$

$= 3(2^{k+1} - 3)$

This is same formula with k replaced by k+1

$\therefore$ if true for $n = k$, it is also true for $n = k+1$

Since true for $n = 1$, by mathematical induction it is true for all positive integers n.

7) $$y = \frac{x^2}{(x-2)(x+1)}$$

i) Asymptotes $x = 2$
$x = -1$
$y = 1$

ii) When $x = 100$

$$y = \frac{100^2}{98 \times 101} = 1.0103$$

so $y \to 1^+$

When $x = -100$

$$y = \frac{(-100)^2}{-102 \times -99} = 0.9903$$

so $y \to 1^-$

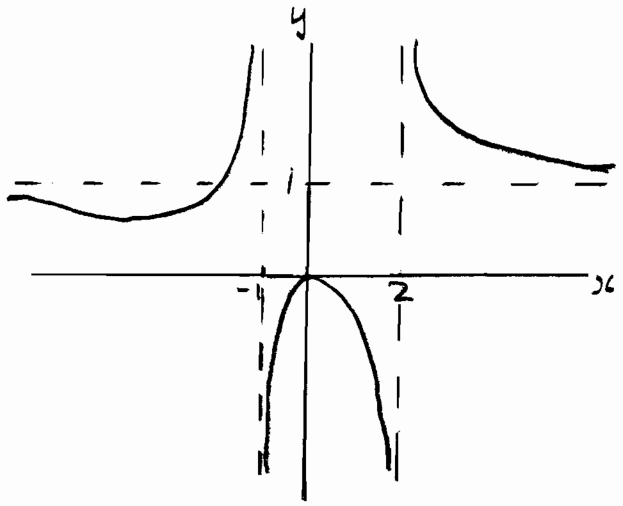
iii) When $x = 0$, $y = 0$

$x = -1.1$ y is $\frac{+}{-\,-}$ +ve

$x = -0.9$ y is $\frac{+}{-\,+}$ -ve

$x = 1.9$ y is $\frac{+}{-\,+}$ -ve

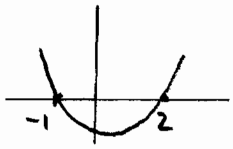
$x = 2.1$ y is $\frac{+}{+\,+}$ +ve



iv) Solve $\frac{x^2}{(x-2)(x+1)} > 0$

Since $x^2 \geq 0$ for all x

We require $(x-2)(x+1) > 0$



graph of $y = (x-2)(x+1)$

Either $x > 2$ or $x < -1$

This is consistent with graph

MEI FP1 JUNE 2006

8) i) Let $x = 2 + j$

$$\Rightarrow x^2 = (2+j)(2+j)$$
$$= 4 + 4j - 1$$
$$= 3 + 4j$$
$$\Rightarrow x^3 = (3+4j)(2+j)$$
$$= 6 + 8j + 3j - 4$$
$$= 2 + 11j$$

Subst for x in

$$2x^3 - 11x^2 + 22x - 15 = 0$$
$$2(2+11j) - 11(3+4j) + 22(2+j) - 15$$
$$= 4 + 22j - 33 - 44j + 44 + 22j - 15$$
$$= 0 + 0j = 0$$

$\therefore$ $x = 2 + j$ is a root of the given equation

ii) $2 - j$

iii) Sum of roots $= \frac{11}{2}$

Let 3rd root be x

$$x + 2 + j + 2 - j = \frac{11}{2}$$
$$x + 4 = \frac{11}{2}$$
$$x = \frac{11}{2} - 4 = \frac{3}{2}$$

Third root $x = \frac{3}{2}$

9) i) $r(r+1)(r+2) - (r-1)r(r+1)$

$$= r(r^2+3r+2) - r(r^2-1)$$
$$= r^3 + 3r^2 + 2r - r^3 + r$$
$$= 3r^2 + 3r = 3r(r+1)$$

ii) $3\sum_{r=1}^{n} r(r+1) = \sum_{r=1}^{n}\left[r(r+1)(r+2) - (r-1)r(r+1)\right]$

r		
1	$1.2.\cancel{3}$	$-\,0.1.2$
2	$\cancel{2.3.4}$	$\cancel{-\,1.2.3}$
3	$\cancel{3.4.5}$	$\cancel{-\,2.3.4}$
4	$\cancel{4.5.6}$	$\cancel{-\,3.4.5}$
$\vdots$		
$n-1$	$\cancel{(n-1)n(n+1)}$	$\cancel{-\,(n-2)(n-1)n}$
n	$n(n+1)(n+2)$	$\cancel{-\,(n-1)n(n+1)}$

$$3\sum_{r=1}^{n} r(r+1) = n(n+1)(n+2) - 0.1.2$$
$$= n(n+1)(n+2)$$
$$\therefore \sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$$

iii) $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$, $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

$$\sum_{r=1}^{n} r(r+1) = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$$
$$= \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1)$$
$$= \frac{1}{6}n(n+1)\left[2n+1+3\right]$$
$$= \frac{1}{6}n(n+1)(2n+4)$$
$$= \frac{1}{3}n(n+1)(n+2) \quad \square$$