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1.

i) Clockwise rotation by θ° anti cl.

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

by 90° anti-clockwise

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

ii) Reflection in $y = x$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

iii)
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

iv) Reflection in x-axis

2.

i) $z = 3 - 2j \qquad w = -4 + j$

$$\frac{z + w}{w} = \frac{3 - 2j + (-4 + j)}{-4 + j}$$

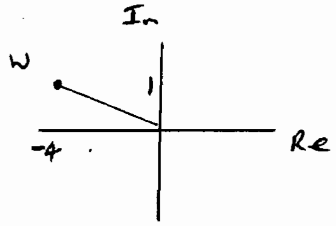
$$= \frac{-1 - j}{-4 + j}$$

$$= \frac{-1 - j}{-4 + j} \times \frac{(-4 - j)}{(-4 - j)}$$

$$= \frac{+4 + 4j + j - 1}{(+4)^2 - j^2}$$

$$= \frac{3 + 5j}{17} = \frac{3}{17} + \frac{5}{17}j$$

ii)



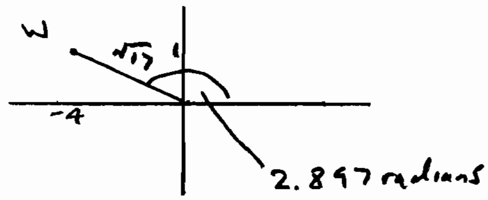
$$|w| = \sqrt{(-4)^2 + 1^2} = \sqrt{17}$$

$$\arg w = \pi - \tan^{-1}\tfrac{1}{4}$$

$$= 2.897 \text{ radians}$$

$$w = \sqrt{17}\left(\cos 2.897 + j \sin 2.897\right)$$

iii)



3. $x^3 + px^2 + qx + 3 = 0$

$\alpha + \beta + \gamma = 4$

$\Rightarrow \quad p = -4$

$$(\alpha + \beta + \gamma)^2 = (\alpha + \beta + \gamma)(\alpha + \beta + \gamma)$$

$$\begin{aligned} &= \alpha^2 + \alpha\beta + \alpha\gamma \\ &\quad + \beta^2 + \alpha\beta + \beta\gamma \\ &\quad + \gamma^2 + \alpha\gamma + \beta\gamma \end{aligned}$$

$$= (\alpha^2 + \beta^2 + \gamma^2) + 2(\alpha\beta + \alpha\gamma + \beta\gamma)$$

$$\therefore 2\Sigma\alpha\beta = (\Sigma\alpha)^2 - (\alpha^2 + \beta^2 + \gamma^2)$$

$$2\Sigma\alpha\beta = 16 - 6 = 10$$

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3) cont

$$\Rightarrow \sum \alpha\beta = \frac{10}{2} = 5$$

$$\Rightarrow q = 5$$

4.

$$\frac{5x}{x^2+4} < x$$

$$\frac{5x}{x^2+4} - x < 0$$

$$\frac{5x - x(x^2+4)}{x^2+4} < 0$$

$$\frac{5x - x^3 - 4x}{x^2+4} < 0$$

$$\frac{x - x^3}{x^2+4} < 0$$

$$\frac{x(1-x^2)}{x^2+4} < 0$$

$$\frac{x(1+x)(1-x)}{x^2+4} < 0$$

Critical values $x = -1, 0, 1$

	-1		0		1	
$\frac{- \; - \; +}{+}$		$\frac{- \; + \; +}{+}$		$\frac{+ \; + \; +}{+}$		$\frac{+ \; + \; -}{+}$
+ve		-ve		+ve		-ve

Solution $-1 < x < 0$

or $x > 1$

5.

$$\frac{3}{(3r-1)(3r+2)} \equiv \frac{1}{3r-1} - \frac{1}{3r+2}$$

r	$\frac{1}{3r-1}$		$\frac{1}{3r+2}$
1	$\frac{1}{2}$	$-$	$\cancel{\frac{1}{5}}$
2	$\cancel{\frac{1}{5}}$	$-$	$\cancel{\frac{1}{8}}$
3	$\cancel{\frac{1}{8}}$	$-$	$\cancel{\frac{1}{11}}$
⋮			
18	$\cancel{\frac{1}{53}}$	$-$	$\cancel{\frac{1}{56}}$
19	$\cancel{\frac{1}{56}}$	$-$	$\cancel{\frac{1}{59}}$
20	$\cancel{\frac{1}{59}}$	$-$	$\frac{1}{62}$

$$\sum_{r=1}^{20} \frac{3}{(3r-1)(3r+2)} = \frac{1}{2} - \frac{1}{62}$$

$$= \frac{31}{62} - \frac{1}{62}$$

$$= \frac{30}{62}$$

$$\therefore \sum_{r=1}^{20} \frac{1}{(3r-1)(3r+2)} = \frac{10}{62}$$

$$= \frac{5}{31}$$

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6. Prove $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

when $n=1$

$$1^3 = 1$$

$$\frac{1}{4}(1)^2(1+1)^2 = \frac{1}{4}\times 4 = 1 \checkmark$$

$\therefore$ true for $n=1$

Assume true for $n=k$

then $\sum_{r=1}^{k} r^3 = \frac{1}{4}k^2(k+1)^2$

$$\Rightarrow \sum_{r=1}^{k+1} r^3 = \frac{1}{4}k^2(k+1)^2 + (k+1)^3$$

$$= \frac{1}{4}(k+1)^2\left[k^2 + 4(k+1)\right]$$

$$= \frac{1}{4}(k+1)^2(k^2+4k+4)$$

$$= \frac{1}{4}(k+1)^2(k+2)^2$$

$$= \frac{1}{4}(k+1)^2((k+1)+1)^2$$

This is same formula with k replaced by $k+1$

So if formula is true for $n=k$, it is true for $n=k+1$.

Since true for $n=1$ then by mathematical induction it is true for all positive integers n

7. $y = \dfrac{(x+9)(3x-8)}{x^2-4}$

i) when $x=0$

$$y = \frac{9(-8)}{-4} = 18$$

Crosses y-axis at $(0, 18)$

when $y=0$ $\quad x=-9$

or $\quad x = \frac{8}{3}$

Crosses x-axis at

$(-9, 0)$ and $\left(\frac{8}{3}, 0\right)$

ii) $y = \dfrac{(x+9)(3x-8)}{(x+2)(x-2)}$

Asymptotes $\quad x=-2$

$x = 2$

$y = 3$

iii) when $x = 100$

$$y = \frac{109 \times 292}{9996} = 3.18$$

As $x \to \infty \quad y \to 3^+$

when $x = -100$

$$y = \frac{-91 \times -308}{9996} = 2.80$$

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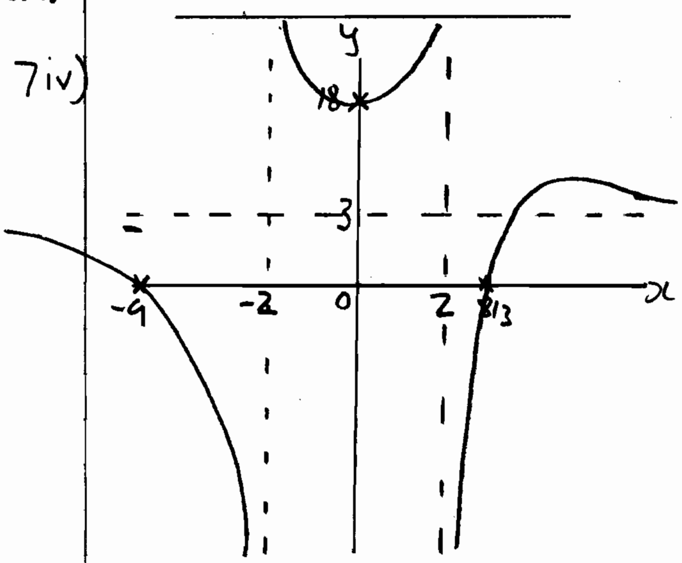
7iii cont $\therefore$ as $x \to -\infty$, $y \to 3^-$

7iv)



8. Two roots $2-j$, $-1+2j$

i) For real coefficients, roots occur in conjugate pairs, so must be at least 4.
$\therefore$ not a cubic.

ii) Other roots $2+j$

and $-1-2j$

$$(z-(2+j))(z-(2-j))$$
$$\times(z-(-1+2j))(z-(-1-2j))=0$$

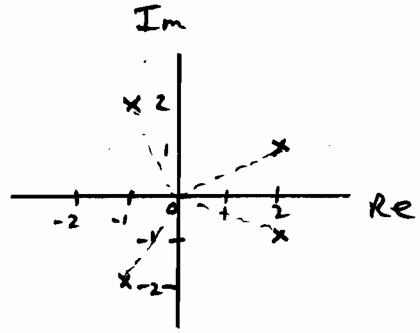
$$\Rightarrow ((z-2)-j)((z-2)+j)$$
$$\times((z+1)-2j)((z+1)+2j)=0$$

$$((z-2)^2+1)((z+1)^2+4)=0$$
$$(z^2-4z+5)(z^2+2z+5)=0$$

$$z^4-4z^3+5z^2$$
$$+2z^3-8z^2+10z$$
$$+5z^2-20z+25=0$$

$$z^4-2z^3+2z^2-10z+25=0$$

iii)



These points would lie on a circle, centre the origin, and radius $\sqrt{2^2+1^2}=\sqrt{5}$

Eqn of circle $|z|=\sqrt{5}$

9. $2x-y=1$

i) $3x+ky=6$

$$\begin{pmatrix}2 & -1\\ 3 & k\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}1\\ 6\end{pmatrix}$$

$$\underline{M}=\begin{pmatrix}2 & -1\\ 3 & k\end{pmatrix}$$

ii) $\underline{M}^{-1}$ does not exist when

$\det \underline{M}=0$

$$\Rightarrow 2k--3=0$$
$$2k+3=0$$
$$k=-\frac{3}{2}$$

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9ii cont)

$$\underline{M}^{-1} = \frac{1}{2k+3}\begin{pmatrix} k & 1 \\ -3 & 2 \end{pmatrix}$$

when it exists

$$\underline{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ b \end{pmatrix}$$

$$\underline{M}^{-1}\underline{M}\begin{pmatrix} x \\ y \end{pmatrix} = \underline{M}^{-1}\begin{pmatrix} 1 \\ b \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \underline{M}^{-1}\begin{pmatrix} 1 \\ b \end{pmatrix}$$

when $k = 5$ and $b = 21$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 5 & 1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 21 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 26 \\ 39 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$x = 2, \quad y = 3$$

9iii) Depending on the value of b, there will either be no solutions (two parallel lines) or an infinite amount of solutions (two eqns representing the same line).

9iv)

A) Non-parallel lines with a unique point of intersection

B) Parallel lines with no point of intersection

C) Both eqns represent the same line, so an infinite amount of solutions

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