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1) $2x(x^2-5) \equiv (x-2)(Ax^2+Bx+C)+D$

$2x^3 - 10x \equiv Ax^3 + Bx^2 + Cx$
$\qquad -2Ax^2 - 2Bx - 2C$
$\qquad + D$

$\Rightarrow A = 2$

$B - 2A = 0$
$B = 2A \qquad \Rightarrow B = 4$

$C - 2B = -10$
$C - 8 = -10$
$\Rightarrow C = -2$

$-2C + D = 0$
$4 + D = 0 \qquad \Rightarrow D = -4$

Solution
$A = 2$
$B = 4$
$C = -2$
$D = -4$

Alternative solution

$2x^3 - 10x \div (x-2)$

$$\begin{array}{r|l} & 2x^2 + 4x - 2 \\ \hline x-2 & 2x^3 \qquad\quad -10x \\ & 2x^3 - 4x^2 \\ \hline & \quad +4x^2 - 10x \\ & \quad +4x^2 - 8x \\ \hline & \qquad\quad -2x \\ & \qquad\quad -2x + 4 \\ \hline & \qquad\qquad -4 \end{array}$$

$2x^3 - 10x =$

$(x-2)(2x^2+4x-2) - 4$

A, B, C, D as above.

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2) $z = \frac{3}{2}$ a root

$\Rightarrow \left(z - \frac{3}{2}\right)$ is a factor

of $2z^3 + 9z^2 + 2z - 30$

so $(2z-3)$ is a factor of

$4z^3 + 18z^2 + 4z - 60$

$$\begin{array}{r|l} & 2z^2 + 12z + 20 \\ \hline 2z-3 & 4z^3 + 18z^2 + 4z - 60 \\ & 4z^3 - 6z^2 \\ \hline & \qquad + 24z^2 + 4z \\ & \qquad + 24z^2 - 36z \\ \hline & \qquad\qquad +40z - 60 \\ & \qquad\qquad +40z - 60 \\ \hline \end{array}$$

$(2z-3)(2z^2+12z+20) = 0$

$\left(z - \frac{3}{2}\right)(z^2 + 6z + 10) = 0$

Other two roots from

$z^2 + 6z + 10 = 0$

$z = \dfrac{-6 \pm \sqrt{36-40}}{2}$

$z = \dfrac{-6 \pm 2j}{2}$

$z = -3 + j$

$z = -3 - j$

I doubled the coefficients to avoid fractions in the long division.

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3) $\underline{N} = \begin{pmatrix} -9 & -2 & -4 \\ 3 & 2 & 2 \\ 5 & 1 & 2 \end{pmatrix}$

$\underline{N}^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ -\frac{7}{2} & p & -6 \end{pmatrix}$

i) $\underline{N}\,\underline{N}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Middle element of top row = 0

$\therefore\ -9(0) - 2(1) - 4(p) = 0$

$-2 - 4p = 0$

$-2 = 4p$

$p = -\frac{1}{2}$

ii) $\underline{N}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -39 \\ 5 \\ 22 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underline{N}^{-1}\begin{pmatrix} -39 \\ 5 \\ 22 \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ -\frac{7}{2} & -\frac{1}{2} & -6 \end{pmatrix}\begin{pmatrix} -39 \\ 5 \\ 22 \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -39 + 0 + 44 \\ -78 + 5 + 66 \\ \frac{273}{2} - \frac{5}{2} - 132 \end{pmatrix} = \begin{pmatrix} 5 \\ -7 \\ 2 \end{pmatrix}$

$x = 5,\ y = -7,\ z = 2$

4) $z_1 = 3 - 2j$

$|z_2| = 5 \quad \arg z_2 = \frac{\pi}{4}$

i) $z_2 = 5\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right)$

$= 5\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)$

$= \frac{5}{\sqrt{2}} + \frac{j}{\sqrt{2}}$

or $\frac{5\sqrt{2}}{2} + \frac{\sqrt{2}}{2}j$

ii) $z_1 + z_2 = \left(3 + \frac{5\sqrt{2}}{2}\right) + \left(-2 + \frac{\sqrt{2}}{2}\right)j$

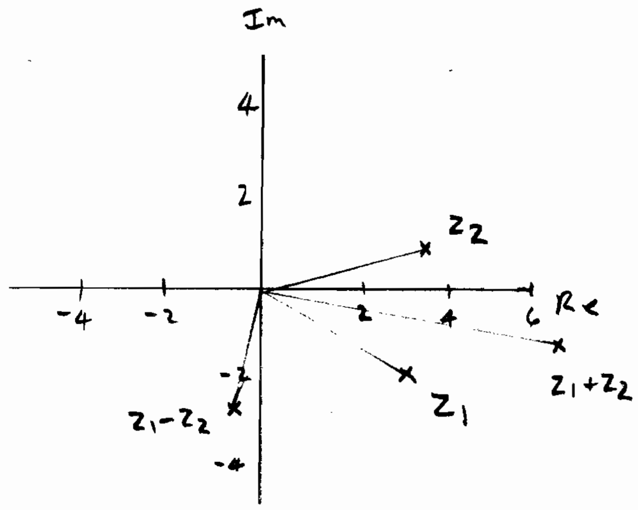
$z_1 - z_2 = \left(3 - \frac{5\sqrt{2}}{2}\right) + \left(-2 - \frac{\sqrt{2}}{2}\right)j$

Approximations

$z_2 \approx 3.5 + 0.7j$

$z_1 + z_2 \approx 6.5 - 1.3j$

$z_1 - z_2 \approx -0.5 - 2.7j$



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5) $\displaystyle\sum_{r=1}^{n} \frac{1}{(4r-3)(4r+1)}$

$$= \frac{1}{4}\sum_{r=1}^{n}\left(\frac{1}{4r-3} - \frac{1}{4r+1}\right)$$

(given in question)

$$= \frac{1}{4}\left[\begin{array}{cccc} r & \frac{1}{4r-3} & - & \frac{1}{4r+1} \\ 1 & \frac{1}{1} & - & \cancel{\frac{1}{5}} \\ 2 & \cancel{\frac{1}{5}} & - & \cancel{\frac{1}{9}} \\ 3 & \cancel{\frac{1}{9}} & - & \cancel{\frac{1}{13}} \\ \vdots & \vdots & & \vdots \\ n-1 & \cancel{\frac{1}{4(n-1)-3}} & - & \cancel{\frac{1}{4(n-1)+1}} \\ n & \cancel{\frac{1}{4n-3}} & - & \frac{1}{4n+1} \end{array}\right]$$

$$= \frac{1}{4}\left[1 - \frac{1}{4n+1}\right]$$

$$= \frac{1}{4}\left[\frac{4n+1-1}{4n+1}\right]$$

$$= \frac{1}{4}\left[\frac{4n}{4n+1}\right]$$

$$= \frac{n}{4n+1}$$

6) $x^3 - 5x^2 + 3x - 6 = 0$

Let $w = \frac{x}{3} + 1$

$\Rightarrow w - 1 = \frac{x}{3}$

$\Rightarrow 3(w-1) = x$

$\Rightarrow (3(w-1))^3 - 5(3(w-1))^2 + 3(3(w-1)) - 6 = 0$

$\Rightarrow 27(w-1)^3 - 45(w-1)^2 + 9(w-1) - 6 = 0$

$27(w^3 - 3w^2 + 3w - 1) - 45(w^2 - 2w + 1) + 9w - 9 - 6 = 0$

$27w^3 - 81w^2 + 81w - 27 - 45w^2 + 90w - 45 + 9w - 15 = 0$

$27w^3 - 126w^2 + 180w - 87 = 0$

$9w^3 - 42w^2 + 60w - 29 = 0$

This is a cubic equation with roots $\frac{\alpha}{3}+1, \frac{\beta}{3}+1, \frac{\gamma}{3}+1$

Alternative solution follows using $\Sigma\alpha$, $\Sigma\alpha\beta$, $\alpha\beta\gamma$

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6 again) $x^3 - 5x^2 + 3x - 6 = 0$

$\Sigma\alpha = 5, \quad \Sigma\alpha\beta = 3, \quad \alpha\beta\gamma = 6$

Require roots $\frac{\alpha}{3}+1, \frac{\beta}{3}+1, \frac{\gamma}{3}+1$

$\Sigma\left(\frac{\alpha}{3}+1\right) = \frac{\alpha}{3}+1+\frac{\beta}{3}+1+\frac{\gamma}{3}+1$

$= \frac{1}{3}(\alpha+\beta+\gamma)+3$

$= \frac{5}{3}+3 = \frac{5+9}{3} = \frac{14}{3}$

$\therefore \Sigma\left(\frac{\alpha}{3}+1\right) = \frac{14}{3}$

$\Sigma\left(\frac{\alpha}{3}+1\right)\left(\frac{\beta}{3}+1\right)$

$= \left(\frac{\alpha}{3}+1\right)\left(\frac{\beta}{3}+1\right)+\left(\frac{\alpha}{3}+1\right)\left(\frac{\gamma}{3}+1\right)$
$+\left(\frac{\beta}{3}+1\right)\left(\frac{\gamma}{3}+1\right)$

$= \frac{\alpha\beta}{9}+\frac{\alpha}{3}+\frac{\beta}{3}+1$
$+\frac{\alpha\gamma}{9}+\frac{\alpha}{3}+\frac{\gamma}{3}+1$
$+\frac{\beta\gamma}{9}+\frac{\beta}{3}+\frac{\gamma}{3}+1$

$= \frac{1}{9}\Sigma\alpha\beta + \frac{2}{3}\Sigma\alpha + 3$

$= \frac{3}{9}+\frac{10}{3}+3 = \frac{20}{3}$

$\therefore \Sigma\left(\frac{\alpha}{3}+1\right)\left(\frac{\beta}{3}+1\right) = \frac{20}{3}$

$\left(\frac{\alpha}{3}+1\right)\left(\frac{\beta}{3}+1\right)\left(\frac{\gamma}{3}+1\right)$

$= \left(\frac{\alpha\beta}{9}+\frac{\alpha}{3}+\frac{\beta}{3}+1\right)\left(\frac{\gamma}{3}+1\right)$

$= \frac{\alpha\beta\gamma}{27}+\frac{\alpha\gamma}{9}+\frac{\beta\gamma}{9}+\frac{\gamma}{3}$
$+\frac{\alpha\beta}{9}+\frac{\alpha}{3}+\frac{\beta}{3}+1$

$= \frac{1}{27}\alpha\beta\gamma + \frac{1}{9}\Sigma\alpha\beta + \frac{1}{3}\Sigma\alpha + 1$

$= \frac{6}{27}+\frac{3}{9}+\frac{5}{3}+1$

$= \frac{2}{9}+\frac{3}{9}+\frac{15}{9}+\frac{9}{9} = \frac{29}{9}$

$\therefore \left(\frac{\alpha}{3}+1\right)\left(\frac{\beta}{3}+1\right)\left(\frac{\gamma}{3}+1\right) = \frac{29}{9}$

Construct eqn with required roots

$w^3 - \frac{14}{3}w^2 + \frac{20}{3}w - \frac{29}{9} = 0$

$9w^3 - 42w^2 + 60w - 29 = 0$

Same answer thankfully!

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7) i) $y = \dfrac{cx^2}{(bx-1)(x+a)}$

For asymptotes $x = \frac{1}{2}$, $x=-2$

$$y = \frac{cx^2}{(2x-1)(x+2)}$$

For asymptote $y = \frac{3}{2}$

$$\frac{c}{2\times 1} = \frac{3}{2} \Rightarrow c=3$$

$\therefore$ $a=2$, $b=2$, $c=3$

$$y = \frac{3x^2}{(2x-1)(x+2)}$$

ii) When $x = 100$

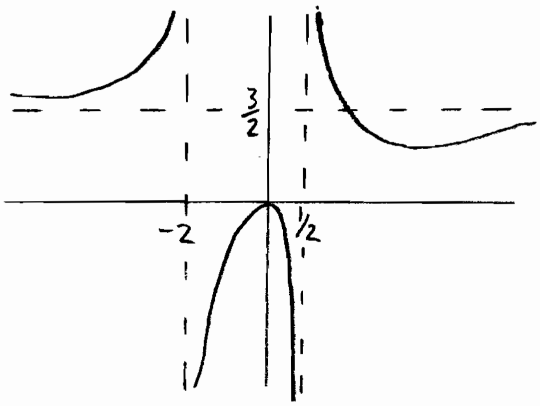
$$y = \frac{3\times 100^2}{199\times 102} \approx 1.48$$

for large +ve x $y \to \frac{3}{2}^-$

When $x = -100$

$$y = \frac{3\times(-100)^2}{(-201)\times(-98)} \approx 1.52$$

for large −ve x $y \to \frac{3}{2}^+$



iii) When $y=1$ $1 = \dfrac{3x^2}{(2x-1)(x+2)}$

$$(2x-1)(x+2) = 3x^2$$

$$2x^2+3x-2 = 3x^2$$

$$0 = x^2-3x+2$$

$$0 = (x-2)(x-1)$$

$\Rightarrow$ $x=1$ or $x=2$

From graph solve

$$\frac{3x^2}{(2x-1)(x+2)} < 1$$

$$-2 < x < \tfrac{1}{2}$$

and $1 < x < 2$

8) i) $\displaystyle\sum_{r=1}^{n}\left[r(r-1)-1\right]$

$$= \sum_{r=1}^{n}\left[r^2 - r - 1\right]$$

$$= \sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r - n$$

$$= \tfrac{1}{6}n(n+1)(2n+1) - \tfrac{1}{2}n(n+1) - n$$

$$= \tfrac{1}{6}n\left[(n+1)(2n+1) - 3(n+1) - 6\right]$$

$$= \tfrac{1}{6}n\left[2n^2+3n+1-3n-3-6\right]$$

$$= \tfrac{1}{6}n\left[2n^2-8\right] = \tfrac{1}{3}n\left[n^2-4\right]$$

$$= \tfrac{1}{3}n(n+2)(n-2)$$

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8ii) Prove $\sum_{r=1}^{n}(r^2-r-1)=\frac{1}{3}n(n+2)(n-2)$

when $n=1$

$1^2-1-1=-1$

$\frac{1}{3}(1)(3)(-1)=-1$ ✓

$\therefore$ true when $n=1$

Assume true for $n=k$

Then $\sum_{r=1}^{k}(r^2-r-1)=\frac{1}{3}k(k+2)(k-2)$

$\Rightarrow \sum_{r=1}^{k+1}(r^2-r-1)$

$=\frac{1}{3}k(k+2)(k-2)+(k+1)^2-(k+1)-1$

$=\frac{1}{3}k(k^2-4)+k^2+2k+1-k-1-1$

$=\frac{1}{3}k(k^2-4)+k^2+k-1$

$=\frac{1}{3}\left[k^3-4k+3k^2+3k-3\right]$

$=\frac{1}{3}\left[k^3+3k^2-k-3\right]$

$=\frac{1}{3}\left[(k+1)(k+3)(k-1)\right]$

(Check: $(k+1)(k-1)=k^2-1$

$(k^2-1)(k+3)$

$=k^3+3k^2-k-3$ ✓)

$=\frac{1}{3}\left[(k+1)((k+1)+2)((k+1)-2)\right]$

This is same formula with k replaced by $k+1$

Thus if true for $n=k$ it is also true for $n=k+1$

Since true for $n=1$, by mathematical induction it is true for all positive integers n

9) i) $\underline{Q}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$

$=\begin{pmatrix}\cos(-90) & -\sin(-90)\\ \sin(-90) & \cos(-90)\end{pmatrix}$

Rotation by $90°$ clockwise about the origin

ii) $\underline{M}=\begin{pmatrix}0 & -1\\ 0 & 1\end{pmatrix}$

A point on $y=2$ is $\begin{pmatrix}x\\ 2\end{pmatrix}$

$\begin{pmatrix}0 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ 2\end{pmatrix}=\begin{pmatrix}-2\\ 2\end{pmatrix}$

P is point $(-2, 2)$

iii) $\begin{pmatrix}0 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}-y\\ y\end{pmatrix}$

$(x, y)\mapsto(-y, y)$

All points map to line $y=-x$

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iv) $y = 6$

since $\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ 6 \end{pmatrix} = \begin{pmatrix} -6 \\ 6 \end{pmatrix}$

v) $\det \underline{M} = 0 \times 1 - 0 \times (-1)$
$= 0 - 0$
$= 0$

$\therefore \underline{M}$ is singular

Under the transformation $\underline{M}$ areas are multiplied by $\det \underline{M}$ ie by 0

Figures with area are mapped onto a line and therefore have 0 area.

vi) $\underline{R} = \underline{Q}\,\underline{M}$

$\underline{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix}$

All points mapped to line $y = x$

vi) Alternatively,

In part (iii) we saw that $\underline{M}$ maps all points to the line $y = -x$

If that is followed by $\underline{Q}$ which we saw in part (i) rotates 90° clockwise about O we would have the line $y = x$