OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE **4756/01**

MATHEMATICS (MEI)

Further Methods for Advanced Mathematics (FP2)

WEDNESDAY 9 JANUARY 2008 Afternoon

Time: 1 hour 30 minutes

Additional materials: Answer Booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Read each question carefully and make sure you know what you have to do before starting your answer.
- Answer **all** the questions in Section A and **one** question from Section B.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **4** printed pages.

Section A (54 marks)

Answer all the questions

1 **(a)** Fig. 1 shows the curve with polar equation $r = a(1 - \cos 2\theta)$ for $0 \leqslant \theta \leqslant \pi$, where a is a positive constant.

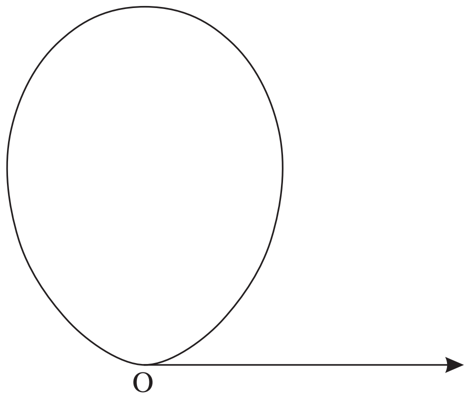

Fig. 1

Find the area of the region enclosed by the curve. [7]

(b) **(i)** Given that $f(x) = \arctan(\sqrt{3} + x)$, find $f'(x)$ and $f''(x)$. [4]

(ii) Hence find the Maclaurin series for $\arctan(\sqrt{3} + x)$, as far as the term in x^2. [4]

(iii) Hence show that, if h is small, $\int_{-h}^{h} x \arctan(\sqrt{3} + x)\,dx \approx \frac{1}{6}h^3$. [3]

2 **(a)** Find the 4th roots of 16j, in the form $re^{j\theta}$ where $r > 0$ and $-\pi < \theta \leqslant \pi$. Illustrate the 4th roots on an Argand diagram. [6]

(b) **(i)** Show that $(1 - 2e^{j\theta})(1 - 2e^{-j\theta}) = 5 - 4\cos\theta$. [3]

Series C and S are defined by

$$C = 2\cos\theta + 4\cos 2\theta + 8\cos 3\theta + \ldots + 2^n \cos n\theta,$$
$$S = 2\sin\theta + 4\sin 2\theta + 8\sin 3\theta + \ldots + 2^n \sin n\theta.$$

(ii) Show that $C = \dfrac{2\cos\theta - 4 - 2^{n+1}\cos(n+1)\theta + 2^{n+2}\cos n\theta}{5 - 4\cos\theta}$, and find a similar expression for S. [9]

3 You are given the matrix $\mathbf{M} = \begin{pmatrix} 7 & 3 \\ -4 & -1 \end{pmatrix}$.

(i) Find the eigenvalues, and corresponding eigenvectors, of the matrix **M**. [8]

(ii) Write down a matrix **P** and a diagonal matrix **D** such that $\mathbf{P}^{-1}\mathbf{MP} = \mathbf{D}$. [2]

(iii) Given that $\mathbf{M}^n = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, show that $a = -\frac{1}{2} + \frac{3}{2} \times 5^n$, and find similar expressions for b, c and d. [8]

Section B (18 marks)

Answer one question

Option 1: Hyperbolic functions

4 **(i)** Given that $k \geqslant 1$ and $\cosh x = k$, show that $x = \pm \ln\left(k + \sqrt{k^2 - 1}\right)$. [5]

(ii) Find $\displaystyle\int_1^2 \frac{1}{\sqrt{4x^2 - 1}}\, dx$, giving the answer in an exact logarithmic form. [5]

(iii) Solve the equation $6 \sinh x - \sinh 2x = 0$, giving the answers in an exact form, using logarithms where appropriate. [4]

(iv) Show that there is no point on the curve $y = 6 \sinh x - \sinh 2x$ at which the gradient is 5. [4]

[Question 5 is printed overleaf.]

Option 2: Investigation of curves

This question requires the use of a graphical calculator.

5 A curve has parametric equations $x = \dfrac{t^2}{1+t^2}$, $y = t^3 - \lambda t$, where λ is a constant.

(i) Use your calculator to obtain a sketch of the curve in each of the cases

$$\lambda = -1, \quad \lambda = 0 \quad \text{and} \quad \lambda = 1.$$

Name any special features of these curves. [5]

(ii) By considering the value of x when t is large, write down the equation of the asymptote. [1]

For the remainder of this question, assume that λ is positive.

(iii) Find, in terms of λ, the coordinates of the point where the curve intersects itself. [3]

(iv) Show that the two points on the curve where the tangent is parallel to the x-axis have coordinates

$$\left(\frac{\lambda}{3+\lambda}, \pm\sqrt{\frac{4\lambda^3}{27}}\right).$$ [4]

Fig. 5 shows a curve which intersects itself at the point (2, 0) and has asymptote $x = 8$. The stationary points A and B have y-coordinates 2 and -2.

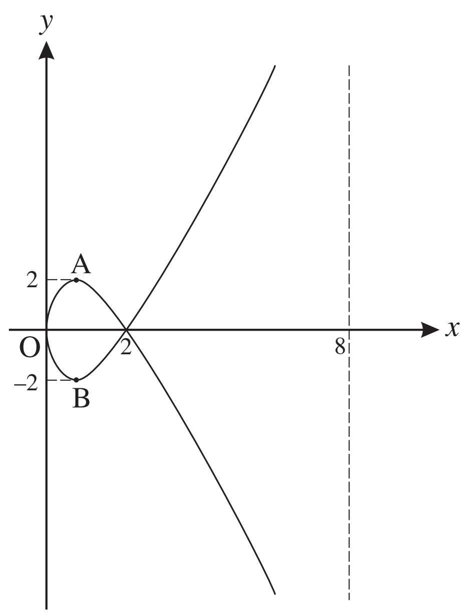

Fig. 5

(v) For the curve sketched in Fig. 5, find parametric equations of the form $x = \dfrac{at^2}{1+t^2}$, $y = b(t^3 - \lambda t)$, where a, λ and b are to be determined. [5]

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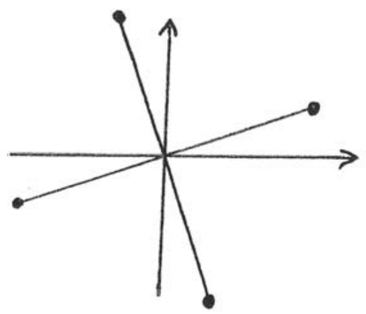
1(a)	Area is $\int_0^{\pi} \frac{1}{2}a^2(1-\cos 2\theta)^2\, d\theta$ $= \int_0^{\pi} \frac{1}{2}a^2\left(1-2\cos 2\theta + \frac{1}{2}(1+\cos 4\theta)\right)d\theta$ $= \frac{1}{2}a^2\left[\frac{3}{2}\theta - \sin 2\theta + \frac{1}{8}\sin 4\theta\right]_0^{\pi}$ $= \frac{3}{4}\pi a^2$	M1 A1 B1 B1B1B1 ft A1 **7**	For $\int (1-\cos 2\theta)^2\, d\theta$ Correct integral expression including limits (may be implied by later work) For $\cos^2 2\theta = \frac{1}{2}(1+\cos 4\theta)$ Integrating $a + b\cos 2\theta + c\cos 4\theta$ *[Max B2 if answer incorrect and no mark has previously been lost]*
(b)(i)	$f'(x) = \frac{1}{1+(\sqrt{3}+x)^2}$ $f''(x) = \frac{-2(\sqrt{3}+x)}{\left(1+(\sqrt{3}+x)^2\right)^2}$	M1 A1 M1 A1 **4**	Applying $\frac{d}{du}\arctan u = \frac{1}{1+u^2}$ or $\frac{dy}{dx} = \frac{1}{\sec^2 y}$ Applying chain (or quotient) rule
(ii)	$f(0) = \frac{1}{3}\pi$ $f'(0) = \frac{1}{4},\quad f''(0) = -\frac{1}{8}\sqrt{3}$ $\arctan(\sqrt{3}+x) = \frac{1}{3}\pi + \frac{1}{4}x - \frac{1}{16}\sqrt{3}\,x^2 + \ldots$	B1 M1 A1A1 ft **4**	Stated; or appearing in series *Accept* 1.05 Evaluating $f'(0)$ or $f''(0)$ For $\frac{1}{4}x$ and $-\frac{1}{16}\sqrt{3}\,x^2$ *ft provided coefficients are non-zero*
(iii)	$\int_{-h}^{h} (\frac{1}{3}\pi x + \frac{1}{4}x^2 - \frac{1}{16}\sqrt{3}\,x^3 + \ldots)\,dx$ $= \left[\frac{1}{6}\pi x^2 + \frac{1}{12}x^3 - \frac{1}{64}\sqrt{3}\,x^4 + \ldots\right]_{-h}^{h}$ $\approx (\frac{1}{6}\pi h^2 + \frac{1}{12}h^3 - \frac{1}{64}\sqrt{3}\,h^4)$ $-(\frac{1}{6}\pi h^2 - \frac{1}{12}h^3 - \frac{1}{64}\sqrt{3}\,h^4)$ $= \frac{1}{6}h^3$	M1 A1 ft A1 ag **3**	Integrating (award if *x* is missed) for $\frac{1}{12}x^3$ Allow ft from $a + \frac{1}{4}x + cx^2$ provided that $a \neq 0$ Condone a proof which neglects h^4

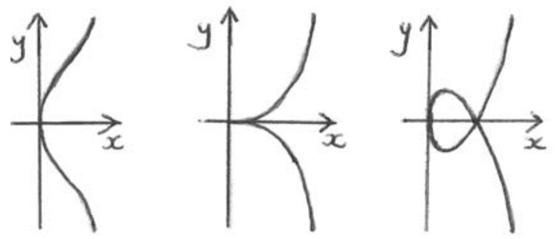
2(a)	4th roots of $16\mathrm{j} = 16\,\mathrm{e}^{\frac{1}{2}\pi\,\mathrm{j}}$ are $r\,\mathrm{e}^{\mathrm{j}\theta}$ where $r = 2$ $\theta = \frac{1}{8}\pi$ $\theta = \frac{\pi}{8} + \frac{2k\pi}{4}$ $\theta = -\frac{7}{8}\pi,\ -\frac{3}{8}\pi,\ \frac{5}{8}\pi$ 	B1 B1 M1 A1 M1 A1 **6**	*Accept* $16^{\frac{1}{4}}$ Implied by at least two correct (ft) further values or stating $k = -2,\ -1,\ (0),\ 1$ Points at vertices of a square centre O *or* 3 correct points (ft) *or* 1 point in each quadrant
(b)(i)	$(1-2\,\mathrm{e}^{\mathrm{j}\theta})(1-2\,\mathrm{e}^{-\mathrm{j}\theta}) = 1 - 2\,\mathrm{e}^{\mathrm{j}\theta} - 2\,\mathrm{e}^{-\mathrm{j}\theta} + 4$ $= 5 - 2(\mathrm{e}^{\mathrm{j}\theta} + \mathrm{e}^{-\mathrm{j}\theta})$ $= 5 - 4\cos\theta$	M1 A1 A1 ag **3**	For $\mathrm{e}^{\mathrm{j}\theta}\mathrm{e}^{-\mathrm{j}\theta} = 1$
	OR $(1 - 2\cos\theta - 2\mathrm{j}\sin\theta)(1 - 2\cos\theta + 2\mathrm{j}\sin\theta)$ M1 $= (1-2\cos\theta)^2 + 4\sin^2\theta$ A1 $= 1 - 4\cos\theta + 4(\cos^2\theta + \sin^2\theta)$ $= 5 - 4\cos\theta$ A1		
(ii)	$C + \mathrm{j}S = 2\,\mathrm{e}^{\mathrm{j}\theta} + 4\,\mathrm{e}^{2\mathrm{j}\theta} + 8\,\mathrm{e}^{3\mathrm{j}\theta} + \ldots + 2^n\,\mathrm{e}^{n\mathrm{j}\theta}$ $= \frac{2\,\mathrm{e}^{\mathrm{j}\theta}\left(1-(2\,\mathrm{e}^{\mathrm{j}\theta})^n\right)}{1-2\,\mathrm{e}^{\mathrm{j}\theta}}$ $= \frac{2\,\mathrm{e}^{\mathrm{j}\theta}(1-2^n\,\mathrm{e}^{n\mathrm{j}\theta})(1-2\,\mathrm{e}^{-\mathrm{j}\theta})}{(1-2\,\mathrm{e}^{\mathrm{j}\theta})(1-2\,\mathrm{e}^{-\mathrm{j}\theta})}$ $= \frac{2\,\mathrm{e}^{\mathrm{j}\theta} - 4 - 2^{n+1}\,\mathrm{e}^{(n+1)\mathrm{j}\theta} + 2^{n+2}\,\mathrm{e}^{n\mathrm{j}\theta}}{5-4\cos\theta}$ $C = \frac{2\cos\theta - 4 - 2^{n+1}\cos(n+1)\theta + 2^{n+2}\cos n\theta}{5-4\cos\theta}$ $S = \frac{2\sin\theta - 2^{n+1}\sin(n+1)\theta + 2^{n+2}\sin n\theta}{5-4\cos\theta}$	M1 M1 A1 M1 A2 M1 A1 ag A1 **9**	Obtaining a geometric series Summing (M0 for sum to infinity) Give A1 for two correct terms in numerator Equating real (or imaginary) parts

3 (i)	Characteristic equation is $(7-\lambda)(-1-\lambda)+12=0$ $\lambda^2-6\lambda+5=0$ $\lambda=1,\ 5$	M1 A1A1	
	When $\lambda=1$, $\begin{pmatrix}7 & 3\\ -4 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$	M1	or $\begin{pmatrix}6 & 3\\ -4 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$ *can be awarded for either eigenvalue*
	$7x+3y=x$ $-4x-y=y$	M1	Equation relating *x* and *y*
	$y=-2x$, eigenvector is $\begin{pmatrix}1\\ -2\end{pmatrix}$	A1	or any (non-zero) multiple
	When $\lambda=5$, $\begin{pmatrix}7 & 3\\ -4 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=5\begin{pmatrix}x\\ y\end{pmatrix}$ $7x+3y=5x$ $-4x-y=5y$	M1	
	$y=-\frac{2}{3}x$, eigenvector is $\begin{pmatrix}3\\ -2\end{pmatrix}$	A1 **8**	*SR* $(\mathbf{M}-\lambda\mathbf{I})\mathbf{x}=\lambda\mathbf{x}$ can earn M1A1A1M0M1A0M1A0
(ii)	$\mathbf{P}=\begin{pmatrix}1 & 3\\ -2 & -2\end{pmatrix}$	B1 ft	B0 if **P** is singular
	$\mathbf{D}=\begin{pmatrix}1 & 0\\ 0 & 5\end{pmatrix}$	B1 ft **2**	For B2, the order must be consistent

(iii)	$\mathbf{M} = \mathbf{P}\,\mathbf{D}\,\mathbf{P}^{-1}$	M1	*May be implied*
	$\mathbf{M}^n = \mathbf{P}\,\mathbf{D}^n\,\mathbf{P}^{-1}$	M1	
	$= \mathbf{P}\begin{pmatrix}1 & 0\\ 0 & 5^n\end{pmatrix}\mathbf{P}^{-1}$	A1 ft	*Dependent on M1M1*
	$= \begin{pmatrix}1 & 3\\ -2 & -2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 5^n\end{pmatrix}\frac{1}{4}\begin{pmatrix}-2 & -3\\ 2 & 1\end{pmatrix}$	B1 ft	For $\mathbf{P}^{-1}$
	$= \begin{pmatrix}1 & 3\times 5^n\\ -2 & -2\times 5^n\end{pmatrix}\frac{1}{4}\begin{pmatrix}-2 & -3\\ 2 & 1\end{pmatrix}$		or $\begin{pmatrix}1 & 3\\ -2 & -2\end{pmatrix}\frac{1}{4}\begin{pmatrix}-2 & -3\\ 2\times 5^n & 5^n\end{pmatrix}$
	$= \frac{1}{4}\begin{pmatrix}-2+6\times 5^n & -3+3\times 5^n\\ 4-4\times 5^n & 6-2\times 5^n\end{pmatrix}$	M1	Obtaining at least one element in a product of three matrices
	$a = -\frac{1}{2} + \frac{3}{2}\times 5^n$	A1 ag	
	$b = -\frac{3}{4} + \frac{3}{4}\times 5^n$ $c = 1 - 5^n$ $\quad d = \frac{3}{2} - \frac{1}{2}\times 5^n$	A2 **8**	Give A1 for one of *b, c, d* correct *SR* If $\mathbf{M}^n = \mathbf{P}^{-1}\,\mathbf{D}^n\,\mathbf{P}$ is used, max marks are M0M1A0B1M1A0A1 (*d* should be correct) *SR* If their **P** is singular, max marks are M1M1A1B0M0

4 (i)	$\frac{1}{2}(e^x + e^{-x}) = k$ $e^{2x} - 2k\,e^x + 1 = 0$ $e^x = \dfrac{2k \pm \sqrt{4k^2 - 4}}{2} = k \pm \sqrt{k^2 - 1}$ $x = \ln(k + \sqrt{k^2 - 1})$ or $\ln(k - \sqrt{k^2 - 1})$ $(k + \sqrt{k^2 - 1})(k - \sqrt{k^2 - 1}) = k^2 - (k^2 - 1) = 1$ $\ln(k - \sqrt{k^2 - 1}) = \ln(\dfrac{1}{k + \sqrt{k^2 - 1}}) = -\ln(k + \sqrt{k^2 - 1})$ $x = \pm\ln(k + \sqrt{k^2 - 1})$	M1 M1 A1 M1 A1 ag **5**	or $\cosh x + \sinh x = e^x$ or $k \pm \sqrt{k^2 - 1} = e^x$ One value sufficient or $\cosh x$ is an even function *(or equivalent)*
(ii)	$\displaystyle\int_1^2 \frac{1}{\sqrt{4x^2 - 1}}\,dx = \left[\frac{1}{2}\operatorname{arcosh} 2x\right]_1^2$ $= \frac{1}{2}(\operatorname{arcosh} 4 - \operatorname{arcosh} 2)$ $= \frac{1}{2}\left(\ln(4 + \sqrt{15}) - \ln(2 + \sqrt{3})\right)$	M1 A1 A1 M1 A1 **5**	For arcosh *or* $\ln(\lambda x + \sqrt{\lambda^2 x^2 - ...})$ or any cosh substitution For $\operatorname{arcosh} 2x$ or $2x = \cosh u$ or $\ln(2x + \sqrt{4x^2 - 1})$ or $\ln(x + \sqrt{x^2 - \frac{1}{4}})$ For $\frac{1}{2}$ or $\int \frac{1}{2}\,du$ Exact numerical logarithmic form
(iii)	$6\sinh x - 2\sinh x\cosh x = 0$ $\cosh x = 3$ (or $\sinh x = 0$) $x = 0$ $x = \pm\ln(3 + \sqrt{8})$	M1 M1 B1 A1 **4**	 Obtaining a value for $\cosh x$ or $x = \ln(3 \pm \sqrt{8})$
	OR $e^{4x} - 6e^{3x} + 6e^x - 1 = 0$ $(e^{2x} - 1)(e^{2x} - 6e^x + 1) = 0$ M2 $x = 0$ B1 $x = \ln(3 \pm \sqrt{8})$ A1		 or $(e^x - e^{-x})(e^x + e^{-x} - 6) = 0$
(iv)	$\dfrac{dy}{dx} = 6\cosh x - 2\cosh 2x$ If $\dfrac{dy}{dx} = 5$ then $6\cosh x - 2(2\cosh^2 x - 1) = 5$ $4\cosh^2 x - 6\cosh x + 3 = 0$ Discriminant $D = 6^2 - 4\times4\times3 = -12$ Since $D < 0$ there are no solutions	B1 M1 M1 A1 **4**	 Using $\cosh 2x = 2\cosh^2 x - 1$ Considering *D*, or completing square, or considering turning point

	OR Gradient $g = 6\cosh x - 2\cosh 2x$ B1 $g' = 6\sinh x - 4\sinh 2x = 2\sinh x(3 - 4\cosh x)$ $= 0$ when $x = 0$ (only) M1 $g'' = 6\cosh x - 8\cosh 2x = -2$ when $x = 0$ M1 Max value $g = 4$ when $x = 0$ So *g* is never equal to 5 A1		Final A1 requires a complete proof showing this is the only turning point

5 (i)	$\lambda=-1$ $\lambda=0$ $\lambda=1$  cusp loop	B1B1B1 B1B1 **5**	 Two different features (cusp, loop, asymptote) correctly identified
(ii)	$x=1$	B1 **1**	
(iii)	Intersects itself when $y=0$ $t=(\pm)\sqrt{\lambda}$ $\left(\dfrac{\lambda}{1+\lambda},\ 0\right)$	M1 A1 A1 **3**	
(iv)	$\dfrac{dy}{dt}=3t^2-\lambda=0$ $t=\pm\sqrt{\dfrac{\lambda}{3}}$ $x=\dfrac{\lambda/3}{1+\lambda/3}=\dfrac{\lambda}{3+\lambda}$ $y=\pm\left((\dfrac{\lambda}{3})^{\frac{3}{2}}-\lambda(\dfrac{\lambda}{3})^{\frac{1}{2}}\right)$ $=\pm\lambda^{\frac{3}{2}}\left(\dfrac{1}{3\sqrt{3}}-\dfrac{1}{\sqrt{3}}\right)=\pm\lambda^{\frac{3}{2}}\left(-\dfrac{2}{3\sqrt{3}}\right)$ $=\pm\sqrt{\dfrac{4\lambda^3}{27}}$	M1 A1 ag M1 A1 ag **4**	 One value sufficient
(v)	From asymptote, $a=8$ From intersection point, $\dfrac{a\lambda}{1+\lambda}=2$ $\lambda=\dfrac{1}{3}$ From maximum point, $b\sqrt{\dfrac{4\lambda^3}{27}}=2$ $b=27$	B1 M1 A1 M1 A1 **5**	