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1a) $\sin y = x$

i) $\cos y . \frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

But $y = \arcsin x$

$\therefore$ derivative of $\arcsin x$

$= \frac{1}{\sqrt{1-x^2}}$

ii) A) $\int_{-1}^{1} \frac{1}{\sqrt{2-x^2}} \, dx$

Now $\int \frac{1}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right)$

$\therefore \int_{-1}^{1} \frac{1}{\sqrt{2-x^2}} \, dx = \left[\arcsin\left(\frac{x}{\sqrt{2}}\right)\right]_{-1}^{1}$

$= \arcsin\left(\frac{1}{\sqrt{2}}\right) - \arcsin\left(-\frac{1}{\sqrt{2}}\right)$

$= \frac{\pi}{4} - -\frac{\pi}{4} = \frac{\pi}{2}$

B) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-2x^2}} \, dx$

Let $u = \sqrt{2}x$

$\Rightarrow \frac{du}{dx} = \sqrt{2}$

$\Rightarrow \frac{1}{\sqrt{2}} du = dx$

when $x = \frac{1}{2}$, $u = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

when $x = -\frac{1}{2}$ $u = -\frac{\sqrt{2}}{2} = \frac{-1}{\sqrt{2}}$

Integral becomes

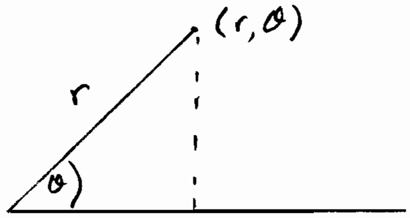
$\frac{1}{\sqrt{2}} \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-u^2}} \, du$

$= \frac{1}{\sqrt{2}} \left[\arcsin u\right]_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}}$

$= \frac{1}{\sqrt{2}} \left[\arcsin\left(\frac{1}{\sqrt{2}}\right) - \arcsin\left(-\frac{1}{\sqrt{2}}\right)\right]$

$= \frac{1}{\sqrt{2}} \left[\frac{\pi}{4} - -\frac{\pi}{4}\right] = \frac{\pi}{2\sqrt{2}}$

1b) $r = \tan\theta$, $0 \le \theta < \frac{\pi}{2}$



$x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow x = \tan\theta\cos\theta = \frac{\sin\theta}{\cos\theta}\cos\theta$

$\Rightarrow x = \sin\theta$

Also $r^2 = \tan^2\theta = \sec^2\theta - 1$

$= \frac{1}{\cos^2\theta} - 1$

$= \frac{1}{\left(\frac{x}{r}\right)^2} - 1$

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1b cont)

$$r^2 = \frac{r^2}{x^2} - 1$$

$$1 = \frac{r^2}{x^2} - r^2$$

$$1 = r^2\left(\frac{1}{x^2} - 1\right)$$

$$1 = r^2\left(\frac{1-x^2}{x^2}\right)$$

$$\frac{x^2}{1-x^2} = r^2$$

$$r^2 = \frac{x^2}{1-x^2}$$

$$y = r\sin\theta$$

$$= \sqrt{\frac{x^2}{1-x^2}} \times x$$

$$= \frac{x}{\sqrt{1-x^2}} \times x$$

$$= \frac{x^2}{\sqrt{1-x^2}}$$

Asymptote is line $x = 1$

2a) i)

$$z = \cos\theta + j\sin\theta$$

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

$$z^n - \frac{1}{z^n} = 2j\sin n\theta$$

ii)

$$\left(z + \frac{1}{z}\right)^4 = z^4 + 4z^3\frac{1}{z} + 6z^2\frac{1}{z^2} + 4z\frac{1}{z^3} + \frac{1}{z^4}$$

$$= z^4 + 4z^2 + 6 + 4\frac{1}{z^2} + \frac{1}{z^4}$$

$$= \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$$

$$\therefore (2\cos\theta)^4 = 2\cos 4\theta + 8\cos 2\theta + 6$$

$$\Rightarrow \cos^4\theta = \frac{2\cos 4\theta + 8\cos 2\theta + 6}{16}$$

$$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$$

iii)

$$16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$$

$$8\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 3$$

$$\cos 4\theta = 8\cos^4\theta - 4\cos 2\theta - 3$$

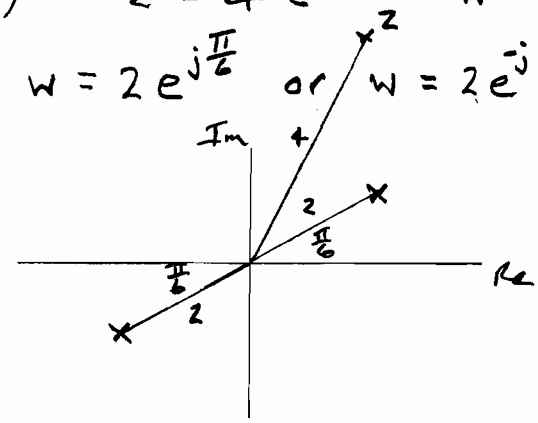
$$\cos 4\theta = 8\cos^4\theta - 4(2\cos^2\theta - 1) - 3$$

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$

2b) i)

$$z = 4e^{j\frac{\pi}{3}} \qquad w^2 = z$$

$$\Rightarrow w = 2e^{j\frac{\pi}{6}} \text{ or } w = 2e^{-j\frac{5\pi}{6}}$$



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2b ii)

$$z = 4e^{j\frac{\pi}{3}}$$

$$z^2 = 4^2 e^{j\frac{2\pi}{3}}$$

$$z^3 = 4^3 e^{j\pi} \quad \text{which is real}$$

so $n = 3$ is smallest value for z^n is real

For z^n purely imaginary

$$n\frac{\pi}{3} = k\pi + \frac{\pi}{2} \quad \text{for } k \in \mathbb{Z}$$

$$\Rightarrow \frac{n}{3} = k + \frac{1}{2}$$

$$\Rightarrow n = 3k + \frac{3}{2}$$

$\Rightarrow n$ not an integer since k is an integer

$$w = 2e^{j\frac{\pi}{6}} \Rightarrow w^3 = 8e^{j\frac{\pi}{2}}$$

$$\Rightarrow w^3 = 8j$$

When $w = 2e^{-j\frac{5\pi}{6}}$

$$w^3 = 8e^{-j\frac{15\pi}{2}} = 8e^{-j\frac{\pi}{2}}$$

$$w^3 = -8j$$

3. $$\underline{M} = \begin{pmatrix} 1 & 2 & 3 \\ -1 & a & 4 \\ 3 & -2 & 2 \end{pmatrix}$$

$$\det \underline{M} = 1(2a+8) - 2(-14) + 3(2-3a)$$

$$= 2a + 8 + 28 + 6 - 9a$$

$$= 42 - 7a$$

For no inverse $\det \underline{M} = 0$

$$\Rightarrow 42 - 7a = 0$$

$$\Rightarrow a = 6$$

Cofactors $$\begin{pmatrix} 2a+8 & -14 & 2-3a \\ 10 & -7 & -8 \\ 8-3a & 7 & a+2 \end{pmatrix}$$

signed cofactors $$\begin{pmatrix} 2a+8 & 14 & 2-3a \\ -10 & -7 & 8 \\ 8-3a & -7 & a+2 \end{pmatrix}$$

Transpose $$\begin{pmatrix} 2a+8 & -10 & 8-3a \\ 14 & -7 & -7 \\ 2-3a & 8 & a+2 \end{pmatrix}$$

$$\underline{M}^{-1} = \frac{1}{42-7a}\begin{pmatrix} 2a+8 & -10 & 8-3a \\ 14 & -7 & -7 \\ 2-3a & 8 & a+2 \end{pmatrix}$$

ii) $a = 0$

$$\frac{1}{42}\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 3 & -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{42}\begin{pmatrix} 8 & -10 & 8 \\ 14 & -7 & -7 \\ 2 & 8 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$x = \frac{8 + 20 + 8}{42} = \frac{36}{42} = \frac{6}{7}$$

$$y = \frac{14 + 14 - 7}{42} = \frac{21}{42} = \frac{1}{2}$$

$$z = \frac{2 - 16 + 2}{42} = -\frac{12}{42} = -\frac{2}{7}$$

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3iii)

$$x + 2y + 3z = 1 \quad (1)$$
$$-x + 6y + 4z = -2 \quad (2)$$
$$3x - 2y + 2z = b \quad (3)$$

(1)+(3) $\quad 4x + 5z = 1 + b$

(2)+3(3) $\quad 8x + 10z = 3b - 2$

For solution $\quad 3b - 2 = 2(1+b)$

$$3b - 2 = 2 + 2b$$
$$3b - 2b = 2 + 2$$
$$b = 4$$

When $b = 4$

$$4x + 5z = 5$$

Let $z = k$

$$4x = 5 - 5z = 5 - 5k$$
$$x = \frac{5-5k}{4}$$

Sub in (1)

$$\frac{5-5k}{4} + 2y + 3k = 1$$
$$2y = 1 - 3k - \frac{(5-5k)}{4}$$
$$8y = 8 - 3k - 5 + 5k$$
$$8y = 3 + 2k$$
$$y = \frac{3+2k}{8}$$

Solution $\quad x = \frac{5-5k}{4}$

$$y = \frac{3+2k}{8}$$
$$z = k$$

for any constant k

This represents a sheaf of planes

Note that this last part could have alternative solutions by choosing $y = k$ or $x = k$.

4.i) Prove $\cosh 2u = 2\sinh^2 u + 1$

$$\sinh u = \frac{1}{2}\left(e^u - e^{-u}\right)$$
$$\Rightarrow \sinh^2 u = \frac{1}{4}\left(e^{2u} - 2 + e^{-2u}\right)$$
$$\Rightarrow 2\sinh^2 u + 1 = \frac{1}{2}\left(e^{2u} - 2 + e^{-2u}\right) + 1$$
$$= \frac{1}{2}\left(e^{2u} + e^{-2u}\right) - 1 + 1$$
$$= \cosh 2u$$

4ii)

$$\cosh y = u \qquad y \geq 0$$
$$\Rightarrow \frac{1}{2}\left(e^y + e^{-y}\right) = u$$
$$\Rightarrow e^y + e^{-y} = 2u$$
$$\Rightarrow e^{2y} + 1 = 2ue^y$$
$$\Rightarrow e^{2y} - 2ue^y + 1 = 0$$

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4ii) cont)

$$\Rightarrow e^y = \frac{2u \pm \sqrt{4u^2-4}}{2}$$

$$\Rightarrow e^y = \frac{2u \pm 2\sqrt{u^2-1}}{2}$$

$$\Rightarrow e^y = u \pm \sqrt{u^2-1}$$

$$\Rightarrow y = \ln\left(u \pm \sqrt{u^2-1}\right)$$

Since product of roots of quadratic is 1

$u + \sqrt{u^2-1}$ is reciprocal of $u - \sqrt{u^2-1}$ and so

$$\ln\left(u - \sqrt{u^2-1}\right) = -\ln\left(u + \sqrt{u^2-1}\right)$$

$$\Rightarrow y = \pm \ln\left(u + \sqrt{u^2-1}\right)$$

Since $y \geqslant 0$

$$y = \ln\left(u + \sqrt{u^2-1}\right)$$

4iii)

$$\int \sqrt{4x^2-1}\,dx$$

Let $2x = \cosh u$

$\Rightarrow 2\frac{dx}{du} = \sinh u$

$\Rightarrow dx = \frac{1}{2}\sinh u\,du$

Integral becomes

$$\int \sqrt{\cosh^2 u - 1}\,.\,\frac{1}{2}\sinh u\,du$$

$$= \int \sinh u\,.\,\frac{1}{2}\sinh u\,du$$

$$= \frac{1}{2}\int \sinh^2 u\,du$$

(Now $\cosh 2u = 1 + 2\sinh^2 u$

so $\frac{\cosh 2u - 1}{2} = \sinh^2 u$)

$$= \frac{1}{4}\int \left(\cosh 2u - 1\right)du$$

$$= \frac{1}{4}\left[\frac{1}{2}\sinh 2u - u\right] + c$$

$$= \frac{1}{8}\sinh 2u - \frac{u}{4} + c$$

$$= \frac{1}{8} \times 2\sinh u \cosh u - \frac{u}{4} + c$$

$$= \frac{1}{4}\sqrt{4x^2-1}\,.\,2x - \frac{\operatorname{arcosh}(2x)}{4} + c$$

$$= \frac{1}{2}x\sqrt{4x^2-1} - \frac{1}{4}\operatorname{arcosh}(2x) + c$$

4iv)

$$\int_{\frac{1}{2}}^{1} \sqrt{4x^2-1}\,dx$$

$$= \left[\frac{1}{2}x\sqrt{4x^2-1} - \frac{1}{4}\operatorname{arcosh}(2x)\right]_{\frac{1}{2}}^{1}$$

$$= \left(\frac{1}{2}\cdot 1\cdot\sqrt{3} - \frac{1}{4}\operatorname{arcosh} 2\right) - \left(\frac{1}{2}\cdot\frac{1}{2}\cdot 0 - \operatorname{arcosh} 1\right)$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{4}\operatorname{arcosh} 2 - (0 - 0)$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{4}\ln\left(2 + \sqrt{3}\right)$$

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