FP2 JANUARY 2011 Q2 COMPLEX NUMBERS

2a) i) $z = \cos\theta + j\sin\theta$

$$z^n + z^{-n} = 2\cos(n\theta)$$

$$z^n - z^{-n} = 2j\sin(n\theta)$$

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ii) $(z^1 + z^{-1})^6 = (2\cos\theta)^6 = 64\cos^6\theta$

but $(z^1+z^{-1})^6$

1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

$$= z^6 + 6z^5\frac{1}{z} + 15z^4\frac{1}{z^2} + 20z^3\frac{1}{z^3} + 15z^2\frac{1}{z^4} + 6z\frac{1}{z^5} + \frac{1}{z^6}$$

$$= \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$$

$$= 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$$

$$\Rightarrow 64\cos^6\theta = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$$

$$\Rightarrow \cos^6\theta = \frac{\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10}{32}$$

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iii) $(z^1 - z^{-1})^6 = (2j\sin\theta)^6 = -64\sin^6\theta$

$$(z^1 - z^{-1})^6 = z^6 - 6z^5\frac{1}{z} + 15z^4\frac{1}{z^2} - 20z^3\frac{1}{z^3} + 15z^2\frac{1}{z^4} - 6z\frac{1}{z^5} + \frac{1}{z^6}$$

$$= \left(z^6 + \frac{1}{z^6}\right) - 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) - 20$$

$$= 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20$$

$$\Rightarrow -64\sin^6\theta = 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20$$

$$\Rightarrow \sin^6\theta = \frac{-\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10}{32}$$

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2aiii) cont) $\therefore \cos^6\theta - \sin^6\theta =$

$$\frac{1}{32}\left(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\right)$$

$$-\frac{1}{32}\left(-\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10\right)$$

$$= \frac{1}{32}\left(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10 + \cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10\right)$$

$$= \frac{1}{32}\left(2\cos 6\theta + 30\cos 2\theta\right)$$

$$= \frac{1}{16}\left(\cos 6\theta + 15\cos 2\theta\right)$$

2b) i) $w = 8e^{j\frac{\pi}{3}}$ $\quad z_1^2 = w$

$z_2^3 = w$

Square Roots of w are

$$\sqrt{8}\, e^{j\frac{\pi}{6}} \quad \text{and} \quad \sqrt{8}\, e^{-j\frac{5\pi}{6}}$$

The one in 3rd quadrant is $2\sqrt{2}\, e^{-j\frac{5\pi}{6}}$

so $z_1 = 2\sqrt{2}\, e^{-j\frac{5\pi}{6}}$

Cubic roots of w are

$$2e^{j\frac{\pi}{9}}, \quad 2e^{j\left(\frac{\pi}{9}+\frac{2\pi}{3}\right)}, \quad 2e^{j\left(\frac{\pi}{9}+\frac{4\pi}{3}\right)}$$

The cubic root in 3rd quadrant is $2e^{j\left(\frac{\pi}{9}+\frac{4\pi}{3}\right)}$

$$= 2e^{j\left(\frac{\pi+12\pi}{9}\right)} = 2e^{j\left(\frac{13\pi}{9}\right)} = 2e^{-j\frac{5\pi}{9}}$$

so $z_2 = 2e^{-j\frac{5\pi}{9}}$

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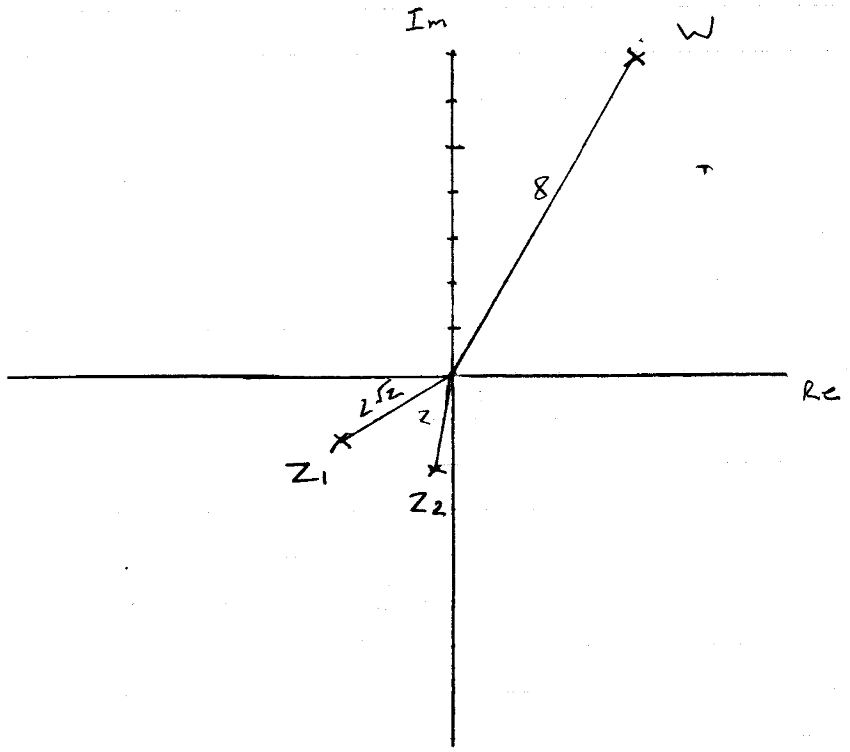
2bi cont)



2bii) $$Z_1 Z_2 = 2\sqrt{2}\, e^{-j\frac{5\pi}{6}} \times 2e^{-j\frac{5\pi}{9}}$$
$$= 4\sqrt{2}\, e^{-j\left(\frac{5\pi}{6}+\frac{5\pi}{9}\right)}$$
$$= 4\sqrt{2}\, e^{-j\left(\frac{15\pi+10\pi}{18}\right)}$$
$$= 4\sqrt{2}\, e^{-j\left(\frac{25\pi}{18}\right)}$$
$$= 4\sqrt{2}\, e^{j\frac{11\pi}{18}}$$

which is located in the second quadrant