FP2 JUNE 2013 Q4 HYPERBOLIC FUNCTIONS

4 i) Prove $\cosh^2 u - \sinh^2 u = 1$

$$\cosh^2 u - \sinh^2 u$$

$$= \left(\tfrac{1}{2}(e^u + e^{-u})\right)^2 - \left(\tfrac{1}{2}(e^u - e^{-u})\right)^2$$

$$= \tfrac{1}{4}\left[(e^u + 2 + e^{-2u}) - (e^u - 2 + e^{-2u})\right]$$

$$= \tfrac{1}{4}\left[4\right] = 1$$

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ii) $y = \operatorname{arsinh} x$

$\Rightarrow \sinh y = x$

$\Rightarrow \cosh y \dfrac{dy}{dx} = 1$ d wrt x

$\Rightarrow \sqrt{1 + \sinh^2 y}\,\dfrac{dy}{dx} = 1$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 + \sinh^2 y}}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 + x^2}}$

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4 ii) cont

$$y = \operatorname{arsinh} x$$

$$\Rightarrow \quad \sinh y = x$$

$$\Rightarrow \quad \tfrac{1}{2}(e^y - e^{-y}) = x$$

$$\Rightarrow \quad e^y - e^{-y} = 2x \qquad \times e^y$$

$$\Rightarrow \quad e^{2y} - 1 = 2xe^y$$

$$\Rightarrow \quad e^{2y} - 2xe^y - 1 = 0$$

$$\Rightarrow \quad e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$$

$$\Rightarrow \quad e^y = \frac{2x \pm 2\sqrt{x^2+1}}{2}$$

$$\Rightarrow \quad e^y = x \pm \sqrt{x^2+1}$$

$$\Rightarrow \quad y = \ln\left(x \pm \sqrt{x^2+1}\right)$$

$$\Rightarrow \quad y = \ln\left(x + \sqrt{x^2+1}\right)$$

Since other value requires log of negative number

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4 iii)

$$\int_0^2 \frac{1}{\sqrt{4+9x^2}}\,dx = \int_0^2 \frac{1}{3\sqrt{\frac{4}{9}+x^2}}\,dx$$

$$= \frac{1}{3}\int_0^2 \frac{1}{\sqrt{\left(\frac{2}{3}\right)^2+x^2}}\,dx$$

$$= \frac{1}{3}\left[\ln\left(x+\sqrt{x^2+\left(\tfrac{2}{3}\right)^2}\right)\right]_0^2$$

$$= \frac{1}{3}\left[\ln\left(2+\sqrt{4+\tfrac{4}{9}}\right) - \ln\left(0+\tfrac{2}{3}\right)\right]$$

$$= \frac{1}{3}\left[\ln\left(2+\sqrt{\tfrac{40}{9}}\right) - \ln\left(\tfrac{2}{3}\right)\right]$$

$$= \frac{1}{3}\left[\ln\left(2+\tfrac{2\sqrt{10}}{3}\right) - \ln\left(\tfrac{2}{3}\right)\right]$$

$$= \frac{1}{3}\ln\left[\frac{2+\frac{2\sqrt{10}}{3}}{2/3}\right]$$

$$= \frac{1}{3}\ln\left[\left(\frac{6+2\sqrt{10}}{3}\right)\times\frac{3}{2}\right]$$

$$= \frac{1}{3}\ln\left(3+\sqrt{10}\right)$$

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* See overleaf for a quicker method *

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4iii again)

$$\int_0^2 \frac{1}{\sqrt{4+9x^2}}\,dx = \frac{1}{3}\int_0^2 \frac{1}{\sqrt{\frac{4}{9}+x^2}}\,dx$$

$$= \frac{1}{3}\int_0^2 \frac{1}{\sqrt{\left(\frac{2}{3}\right)^2+x^2}}\,dx$$

$$= \frac{1}{3}\left[\operatorname{arsinh}\left(\frac{x}{\frac{2}{3}}\right)\right]_0^2$$

$$= \frac{1}{3}\left[\operatorname{arsinh}\left(\frac{3x}{2}\right)\right]_0^2$$

$$= \frac{1}{3}\left[\operatorname{arsinh} 3 - \operatorname{arsinh} 0\right]$$

$$= \frac{1}{3}\operatorname{arsinh} 3 - 0$$

$$= \frac{1}{3}\ln\left(3+\sqrt{3^2+1}\right)$$

$$= \frac{1}{3}\ln\left(3+\sqrt{10}\right)$$

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4 iv)

$$\int_0^1 \frac{1}{\sqrt{1+x^2}} \operatorname{arsinh} x \, dx$$

Let $u = \operatorname{arsinh} x$

$$\frac{du}{dx} = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow \quad du = \frac{1}{\sqrt{1+x^2}} \, dx$$

when $x = 1$ $u = \ln(1+\sqrt{2})$

when $x = 0$ $u = 0$

Integral becomes

$$\int_0^{\ln(1+\sqrt{2})} u \, du$$

$$= \left[\frac{u^2}{2} \right]_0^{\ln(1+\sqrt{2})}$$

$$= \frac{(\ln(1+\sqrt{2}))^2}{2} - 0$$

$$= \frac{(\ln(1+\sqrt{2}))^2}{2}$$