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3a) i) $\underline{A} = \begin{pmatrix} 6 & -3 \\ 4 & -1 \end{pmatrix}$

Characteristic equation $\det(\underline{A} - \lambda \underline{I}) = 0$

$$\det \begin{pmatrix} 6-\lambda & -3 \\ 4 & -1-\lambda \end{pmatrix} = 0$$

$$(6-\lambda)(-1-\lambda) - 4(-3) = 0$$

$$-6 + \lambda - 6\lambda + \lambda^2 + 12 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 3)(\lambda - 2) = 0$$

$\Rightarrow \lambda = 3$ or $\lambda = 2$ eigen values

If $(\underline{A} - \lambda \underline{I})\underline{s} = \underline{0}$ when $\lambda = 3$

$$\begin{pmatrix} 6-3 & -3 \\ 4 & -1-3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -3 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow 3x - 3y = 0$

$\Rightarrow x = y$ suitable eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
with eigenvalue $\lambda = 3$

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3ai) cont) If $(\underline{A} - \lambda \underline{I})\underline{s} = \underline{0}$ when $\lambda = 2$

$$\begin{pmatrix} 6-2 & -3 \\ 4 & -1-2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{pmatrix} 4 & -3 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\Rightarrow \quad 4x - 3y = 0$$

$$\Rightarrow \quad 4x = 3y$$

suitable eigenvector is $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

with eigenvalue $\lambda = 2$

3a)ii)

If $\underline{A} = \underline{P}\,\underline{D}\,\underline{P}^{-1}$

then $\underline{P} = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$ and $\underline{D} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$

Alternatively

$\underline{P} = \begin{pmatrix} 3 & 1 \\ 4 & 1 \end{pmatrix}$ and $\underline{D} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

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3b) i)

$$\lambda^3 - 4\lambda^2 - 3\lambda - 10 = 0$$

Show $\lambda = 5$ is an eigenvalue

$$5^3 - 4 \times 5^2 - 3 \times 5 - 10$$

$$= 125 - 100 - 15 - 10 = 0$$

$\lambda = 5$ satisfies characteristic equation and so is an eigenvalue

$$\begin{array}{r|l} & \lambda^2 + \lambda + 2 \\ \hline \lambda - 5 & \lambda^3 - 4\lambda^2 - 3\lambda - 10 \\ & \underline{\lambda^3 - 5\lambda^2} \\ & \qquad \lambda^2 - 3\lambda \\ & \qquad \underline{\lambda^2 - 5\lambda} \\ & \qquad\qquad 2\lambda - 10 \\ & \qquad\qquad \underline{2\lambda - 5} \end{array}$$

$$(\lambda - 5)(\lambda^2 + \lambda + 2) = 0$$

For quadratic term discriminant $b^2 - 4ac = 1 - 8 < 0$

$\therefore$ no solutions to $\lambda^2 + \lambda + 2 = 0$

and so no other real eigenvalues.

3b) ii) Given $\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$ is an eigenvector with eigenvalue $\lambda = 5$

$$\therefore \quad \underline{B} \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} = 5 \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -10 \\ 5 \\ 20 \end{pmatrix}$$

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3bii) cont

$$\underline{B}^2\begin{pmatrix}4\\-2\\-8\end{pmatrix} = -2\,\underline{B}^2\begin{pmatrix}-2\\1\\4\end{pmatrix}$$

$$= -2\,\underline{B}\,\underline{B}\begin{pmatrix}-2\\1\\4\end{pmatrix}$$

$$= -2\,\underline{B}\begin{pmatrix}-10\\5\\20\end{pmatrix}$$

$$= -10\,\underline{B}\begin{pmatrix}-2\\1\\4\end{pmatrix}$$

$$= -10\begin{pmatrix}-10\\5\\20\end{pmatrix}$$

$$= \begin{pmatrix}100\\-50\\-200\end{pmatrix}$$

$$\underline{B}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-20\\10\\40\end{pmatrix}$$

$$\underline{B}\begin{pmatrix}x\\y\\z\end{pmatrix} = 5\begin{pmatrix}-4\\2\\8\end{pmatrix}$$

But $\underline{B}\begin{pmatrix}-4\\2\\8\end{pmatrix} = 5\begin{pmatrix}-4\\2\\8\end{pmatrix}$ (linear multiple of original eigenvector)

$\Rightarrow x = -4,\ y = 2,\ z = 8$

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3b)iii) By Cayley–Hamilton theorem,

$\underline{B}$ satisfies its own characteristic equation

$$\therefore \quad \underline{B}^3 - 4\underline{B}^2 - 3\underline{B} - 10\underline{I} = \underline{0}$$

$$\Rightarrow \quad \underline{B}^4 - 4\underline{B}^3 - 3\underline{B}^2 - 10\underline{B} = \underline{0}$$

$$\Rightarrow \quad \underline{B}^4 = 4\underline{B}^3 + 3\underline{B}^2 + 10\underline{B}$$

$$\Rightarrow \quad \underline{B}^4 = 4\left(4\underline{B}^2 + 3\underline{B} + 10\underline{I}\right) + 3\underline{B}^2 + 10\underline{B}$$

$$\Rightarrow \quad \underline{B}^4 = 16\underline{B}^2 + 12\underline{B} + 40\underline{I} + 3\underline{B}^2 + 10\underline{B}$$

$$\Rightarrow \quad \underline{B}^4 = 19\underline{B}^2 + 22\underline{B} + 40\underline{I}$$