FP2 JANUARY 2007 Q4 HYPERBOLIC FUNCTIONS

4 i) Show $\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2-1}\right)$

Let $y = \operatorname{arcosh} x$

$$\Rightarrow \cosh y = x$$

$$\tfrac{1}{2}\left(e^{y}+e^{-y}\right) = x$$

$$e^{y} + e^{-y} = 2x$$

$$e^{2y} + 1 = 2xe^{y} \qquad (\times\ e^{y})$$

$$e^{2y} - 2xe^{y} + 1 = 0$$

$$e^{y} = \frac{2x \pm \sqrt{4x^2-4}}{2} \qquad \text{(using quadratic formula)}$$

$$e^{y} = \frac{2x \pm 2\sqrt{x^2-1}}{2}$$

$$e^{y} = x \pm \sqrt{x^2-1}$$

Since $y > 0$, $e^{y} > 1$ so $e^{y} = x + \sqrt{x^2-1}$

$$\Rightarrow y = \ln\left(x + \sqrt{x^2-1}\right)$$

4ii) $$\int_{2.5}^{3.9} \frac{1}{\sqrt{4x^2-9}}\,dx$$

Standard integral $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right) + c$

$\left(\text{or } \operatorname{arcosh}\left(\frac{x}{a}\right) + c\right)$

4ii) cont)

$$\int_{2.5}^{3.9} \frac{1}{\sqrt{4x^2-9}}\,dx = \int_{2.5}^{3.9} \frac{1}{\sqrt{4(x^2-\frac{9}{4})}}$$

$$= \frac{1}{2}\int_{2.5}^{3.9} \frac{1}{\sqrt{x^2-(\frac{3}{2})^2}}\,dx$$

$$= \frac{1}{2}\left[\ln\left(x+\sqrt{x^2-(3/2)^2}\right)\right]_{2.5}^{3.9}$$

$$= \frac{1}{2}\left[\ln\left(3.9+\sqrt{3.9^2-1.5^2}\right) - \ln\left(2.5+\sqrt{2.5^2-1.5^2}\right)\right]$$

$$= \frac{1}{2}\left[\ln 7.5 - \ln 4.5\right]$$

$$= \frac{1}{2}\ln\left(\frac{7.5}{4.5}\right) = \frac{1}{2}\ln\left(\frac{75}{45}\right) = \frac{1}{2}\ln\left(\frac{5}{3}\right)$$

4iii)

$$y = \frac{\cosh x}{2+\sinh x}$$

$$\frac{dy}{dx} = \frac{(2+\sinh x)\sinh x - \cosh x \times \cosh x}{(2+\sinh x)^2}$$

$$\frac{dy}{dx} = \frac{2\sinh x + \sinh^2 x - \cosh^2 x}{(2+\sinh x)^2}$$

$$\frac{dy}{dx} = \frac{2\sinh x - 1}{(2+\sinh x)^2}$$ (since $\cosh^2 x - \sinh^2 x = 1$)

4 iii) cont) If gradient $= \frac{1}{9}$ then

$$\frac{2\sinh x - 1}{(2+\sinh x)^2} = \frac{1}{9}$$

$$\Rightarrow \quad 9(2\sinh x - 1) = (2+\sinh x)^2$$

$$18\sinh x - 9 = 4 + 4\sinh x + \sinh^2 x$$

$$0 = \sinh^2 x - 14\sinh x + 13$$

$$0 = (\sinh x - 13)(\sinh x - 1)$$

$$\Rightarrow \quad \sinh x = 13 \quad \text{or} \quad \sinh x = 1$$

$$\Rightarrow \quad x = \operatorname{arsinh} 13 \quad \text{or} \quad x = \operatorname{arsinh} 1$$

Using $\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right)$

$$x = \ln\left(13 + \sqrt{169+1}\right) \quad \text{or} \quad x = \ln\left(1 + \sqrt{1+1}\right)$$

$$x = \ln\left(13 + \sqrt{170}\right) \quad \text{or} \quad x = \ln\left(1 + \sqrt{2}\right)$$

Find y coords

when $\sinh x = 13$

$\cosh x = \sqrt{1+13^2}$

$\cosh x = \sqrt{170}$

$$y = \frac{\cosh x}{2+\sinh x} = \frac{\sqrt{170}}{2+13}$$

when $\sinh x = 1$

$\cosh x = \sqrt{1+1^2}$

$\cosh x = \sqrt{2}$

$$y = \frac{\sqrt{2}}{2+1}$$

Point is $\left(\ln(13+\sqrt{170}), \frac{\sqrt{170}}{15}\right)$

Point is $\left(\ln(1+\sqrt{2}), \frac{\sqrt{2}}{3}\right)$