FP2 JANUARY 2008 Q4 HYPERBOLIC FUNCTIONS

4 (i)

$k \geq 1$ Let $\cosh x = k$

$$\Rightarrow \frac{1}{2}\left(e^x + e^{-x}\right) = k$$

$$e^x + e^{-x} = 2k$$

$$e^{2x} + 1 = 2ke^x \qquad (\times e^x)$$

$$e^{2x} - 2ke^x + 1 = 0$$

$$\Rightarrow e^x = \frac{2k \pm \sqrt{4k^2 - 4}}{2}$$

$$\Rightarrow e^x = \frac{2k \pm 2\sqrt{k^2 - 1}}{2}$$

$$\Rightarrow e^x = k \pm \sqrt{k^2 - 1}$$

$$\Rightarrow x = \ln\left(k \pm \sqrt{k^2-1}\right)$$

Now $\left(k + \sqrt{k^2-1}\right)\left(k - \sqrt{k^2-1}\right) = k^2 - (k^2-1) = 1$

$$\therefore k - \sqrt{k^2-1} = \frac{1}{k + \sqrt{k^2-1}}$$

$$\therefore \ln\left(k - \sqrt{k^2-1}\right) = \ln\left(\frac{1}{k+\sqrt{k^2-1}}\right) = -\ln\left(k + \sqrt{k^2-1}\right)$$

$$\Rightarrow x = \pm \ln\left(k + \sqrt{k^2-1}\right)$$

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4 (ii)

$$\int_1^2 \frac{1}{\sqrt{4x^2-1}}\,dx = \int_1^2 \frac{1}{\sqrt{4\left(x^2 - \frac{1}{4}\right)}}\,dx = \frac{1}{2}\int_1^2 \frac{1}{\sqrt{x^2 - \left(\frac{1}{2}\right)^2}}\,dx$$

4 ii cont) Now $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \text{arcosh}\left(\frac{x}{a}\right) = \ln\left(x + \sqrt{x^2-a^2}\right) + c$

$$\therefore \frac{1}{2}\int_1^2 \frac{1}{\sqrt{x^2-\left(\frac{1}{2}\right)^2}}\,dx = \frac{1}{2}\left[\ln\left(x+\sqrt{x^2-\left(\tfrac{1}{2}\right)^2}\right)\right]_1^2$$

$$= \frac{1}{2}\left[\ln\left(2+\sqrt{4-\tfrac{1}{4}}\right) - \ln\left(1+\sqrt{1-\tfrac{1}{4}}\right)\right]$$

$$= \frac{1}{2}\left[\ln\left(2+\sqrt{\tfrac{15}{4}}\right) - \ln\left(1+\sqrt{\tfrac{3}{4}}\right)\right]$$

$$= \frac{1}{2}\left[\ln\left(\frac{4+\sqrt{15}}{2}\right) - \ln\left(\frac{2+\sqrt{3}}{2}\right)\right]$$

$$= \frac{1}{2}\ln\left(\frac{\frac{4+\sqrt{15}}{2}}{\frac{2+\sqrt{3}}{2}}\right) = \frac{1}{2}\ln\left(\frac{4+\sqrt{15}}{2+\sqrt{3}}\right)$$

4(iii)

$$6\sinh x - \sinh 2x = 0$$

$$6\sinh x - 2\sinh x\cosh x = 0$$

$$2\sinh x(3-\cosh x) = 0$$

$$\Rightarrow \sinh x = 0 \quad \text{or} \quad 3 - \cosh x = 0$$

$$\Rightarrow x = 0 \qquad\qquad 3 = \cosh x$$

$$x = \text{arcosh}\, 3$$

$$x = \pm\ln(3+\sqrt{8})$$

4(iv) $y = 6\sinh x - \sinh 2x$

$$\frac{dy}{dx} = 6\cosh x - 2\cosh 2x$$

Assume $\frac{dy}{dx} = 5$

then $6\cosh x - 2\cosh 2x = 5$

$$6\cosh x - 2(2\cosh^2 x - 1) = 5$$

$$6\cosh x - 4\cosh^2 x + 2 = 5$$

$$0 = 4\cosh^2 x - 6\cosh x + 3 = 0$$

$$\cosh x = \frac{6 \pm \sqrt{36 - 48}}{8}$$

$$\cosh x = \frac{6 \pm \sqrt{-12}}{8}$$ Discriminant < 0

No solution so assumption was incorrect

$\frac{dy}{dx}$ cannot be equal to 5