FP2 JUNE 2009 Q4 HYPERBOLIC FUNCTIONS

4 i)

$$\cosh 2u = \tfrac{1}{2}(e^{2u} + e^{-2u})$$

$$\begin{aligned}
2\cosh^2 u - 1 &= 2\left(\tfrac{1}{2}(e^{u}+e^{-u})\right)^2 - 1 \\
&= 2\left(\tfrac{1}{4}(e^{2u}+2+e^{-2u})\right) - 1 \\
&= \tfrac{1}{2}(e^{2u}+2+e^{-2u}) - 1 \\
&= \tfrac{1}{2}(e^{2u}+e^{-2u}) + 1 - 1 \\
&= \tfrac{1}{2}(e^{2u}+e^{-2u}) \\
&= \cosh 2u
\end{aligned}$$

ii) Prove $\operatorname{arsinh} y = \ln\left(y + \sqrt{y^2+1}\right)$

Let $x = \operatorname{arsinh} y$

$$\begin{aligned}
\Rightarrow\quad & \sinh x = y \\
\Rightarrow\quad & \tfrac{1}{2}(e^{x} - e^{-x}) = y \\
\Rightarrow\quad & e^{x} - e^{-x} = 2y \\
\Rightarrow\quad & e^{2x} - 1 = 2ye^{x} \qquad (\times e^{x}) \\
\Rightarrow\quad & e^{2x} - 2ye^{x} - 1 = 0 \\
\Rightarrow\quad & e^{x} = \frac{2y \pm \sqrt{4y^2+4}}{2} \\
\Rightarrow\quad & e^{x} = \frac{2y \pm 2\sqrt{y^2+1}}{2} \\
\Rightarrow\quad & e^{x} = y \pm \sqrt{y^2+1} \\
\Rightarrow\quad & e^{x} = y + \sqrt{y^2+1} \qquad \text{since } e^{x} > 0
\end{aligned}$$

4ii) cont)

$$\Rightarrow \quad x = \ln\left(y + \sqrt{y^2+1}\right)$$

$$\Rightarrow \quad \operatorname{arsinh} y = \ln\left(y + \sqrt{y^2+1}\right)$$

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4 iii)

$$\int \sqrt{x^2+4}\, dx$$

Let $x = 2\sinh u$

$\dfrac{dx}{du} = 2\cosh u$

$dx = 2\cosh u \, du$

$\dfrac{x}{2} = \sinh u$

$$= \int \sqrt{4\sinh^2 u + 4} \times 2\cosh u \, du$$

$$= \int 2\sqrt{\sinh^2 + 1} \times 2\cosh u \, du$$

$$= \int 2\sqrt{\cosh^2 u} \times 2\cosh u \, du$$

$$= \int 4\cosh^2 u \, du$$

$$= \int 2\left(\cosh 2u + 1\right) du$$

$$= \int \left(2\cosh 2u + 2\right) du$$

$$= \sinh 2u + 2u + c$$

$$= 2\sinh u \cosh u + 2u + c$$

$$= 2\sqrt{1+\sinh^2 u} \times \sinh u + 2u + c$$

$$= 2\sqrt{1+\left(\frac{x}{2}\right)^2} \times \frac{x}{2} + 2\operatorname{arsinh}\left(\frac{x}{2}\right) + c$$

$$= x\sqrt{\frac{4+x^2}{4}} + 2\operatorname{arsinh}\left(\frac{x}{2}\right) + c$$

$$= \frac{1}{2}x\sqrt{4+x^2} + 2\operatorname{arsinh}\left(\frac{x}{2}\right) + c$$

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4 iv) $t^2+2t+5 = (t+1)^2+5-1 = (t+1)^2+4$

$$\int_{-1}^{1}\sqrt{t^2+2t+5}\,dt = \int_{-1}^{1}\sqrt{(t+1)^2+4}\,dt$$

Using part (iii) $$= \left[\tfrac{1}{2}(t+1)\sqrt{4+(t+1)^2} + 2\operatorname{arsinh}\left(\tfrac{t+1}{2}\right)\right]_{-1}^{1}$$

$$= \left(\tfrac{1}{2}\times 2\sqrt{4+4} + 2\operatorname{arsinh}(1)\right) - \left(0 + 2\operatorname{arsinh} 0\right)$$

$$= 2\sqrt{2} + 2\ln\left(1+\sqrt{1^2+1}\right) - 0 - 0$$

$$= 2\sqrt{2} + 2\ln(1+\sqrt{2})$$

$$= 2\left(\sqrt{2} + \ln(1+\sqrt{2})\right)$$ as required