RECOGNISING ACHIEVEMENT

GCE

Mathematics (MEI)

Advanced GCE

Unit **4756:** Further Methods for Advanced Mathematics

Mark Scheme for January 2013

Oxford Cambridge and RSA Examinations

This mark scheme is published as an aid to teachers and students, to indicate the requirements of the examination. It shows the basis on which marks were awarded by examiners. It does not indicate the details of the discussions which took place at an examiners' meeting before marking commenced.

All examiners are instructed that alternative correct answers and unexpected approaches in candidates' scripts must be given marks that fairly reflect the relevant knowledge and skills demonstrated.

Mark schemes should be read in conjunction with the published question papers and the report on the examination.

OCR will not enter into any discussion or correspondence in connection with this mark scheme.

Annotations

Annotation	Meaning
✓and ✗	
BOD	Benefit of doubt
FT	Follow through
ISW	Ignore subsequent working
M0, M1	Method mark awarded 0, 1
A0, A1	Accuracy mark awarded 0, 1
B0, B1	Independent mark awarded 0, 1
SC	Special case
^	Omission sign
MR	Misread
Highlighting	
Other abbreviations in mark scheme	**Meaning**
E1	Mark for explaining
U1	Mark for correct units
G1	Mark for a correct feature on a graph
M1 dep*	Method mark dependent on a previous mark, indicated by *
cao	Correct answer only
oe	Or equivalent
rot	Rounded or truncated
soi	Seen or implied
www	Without wrong working

Subject-specific Marking Instructions

a Annotations should be used whenever appropriate during your marking.

The A, M and B annotations must be used on your standardisation scripts for responses that are not awarded either 0 or full marks. It is vital that you annotate standardisation scripts fully to show how the marks have been awarded.

For subsequent marking you must make it clear how you have arrived at the mark you have awarded.

b An element of professional judgement is required in the marking of any written paper. Remember that the mark scheme is designed to assist in marking incorrect solutions. Correct *solutions* leading to correct answers are awarded full marks but work must not be judged on the answer alone, and answers that are given in the question, especially, must be validly obtained; key steps in the working must always be looked at and anything unfamiliar must be investigated thoroughly.

Correct but unfamiliar or unexpected methods are often signalled by a correct result following an *apparently* incorrect method. Such work must be carefully assessed. When a candidate adopts a method which does not correspond to the mark scheme, award marks according to the spirit of the basic scheme; if you are in any doubt whatsoever (especially if several marks or candidates are involved) you should contact your Team Leader.

c The following types of marks are available.

M
A suitable method has been selected and *applied* in a manner which shows that the method is essentially understood. Method marks are not usually lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, eg by substituting the relevant quantities into the formula. In some cases the nature of the errors allowed for the award of an M mark may be specified.

A
Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated Method mark is earned (or implied). Therefore M0 A1 cannot ever be awarded.

B
Mark for a correct result or statement independent of Method marks.

E
A given result is to be established or a result has to be explained. This usually requires more working or explanation than the establishment of an unknown result.

Unless otherwise indicated, marks once gained cannot subsequently be lost, eg wrong working following a correct form of answer is ignored. Sometimes this is reinforced in the mark scheme by the abbreviation isw. However, this would not apply to a case where a candidate passes through the correct answer as part of a wrong argument.

d When a part of a question has two or more ‘method’ steps, the M marks are in principle independent unless the scheme specifically says otherwise; and similarly where there are several B marks allocated. (The notation ‘dep *’ is used to indicate that a particular mark is dependent on an earlier, asterisked, mark in the scheme.) Of course, in practice it may happen that when a candidate has once gone wrong in a part of a question, the work from there on is worthless so that no more marks can sensibly be given. On the other hand, when two or more steps are successfully run together by the candidate, the earlier marks are implied and full credit must be given.

e The abbreviation ft implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A and B marks are given for correct work only – differences in notation are of course permitted. A (accuracy) marks are not given for answers obtained from incorrect working. When A or B marks are awarded for work at an intermediate stage of a solution, there may be various alternatives that are equally acceptable. In such cases, exactly what is acceptable will be detailed in the mark scheme rationale. If this is not the case please consult your Team Leader.

Sometimes the answer to one part of a question is used in a later part of the same question. In this case, A marks will often be ‘follow through’. In such cases you must ensure that you refer back to the answer of the previous part question even if this is not shown within the image zone. You may find it easier to mark follow through questions candidate-by-candidate rather than question-by-question.

f Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise. Candidates are expected to give numerical answers to an appropriate degree of accuracy, with 3 significant figures often being the norm. Small variations in the degree of accuracy to which an answer is given (eg 2 or 4 significant figures where 3 is expected) should not normally be penalised, while answers which are grossly over- or under-specified should normally result in the loss of a mark. The situation regarding any particular cases where the accuracy of the answer may be a marking issue should be detailed in the mark scheme rationale. If in doubt, contact your Team Leader.

g Rules for replaced work

If a candidate attempts a question more than once, and indicates which attempt he/she wishes to be marked, then examiners should do as the candidate requests.

If there are two or more attempts at a question which have not been crossed out, examiners should mark what appears to be the last (complete) attempt and ignore the others.

NB Follow these maths-specific instructions rather than those in the assessor handbook.

h For a *genuine* misreading (of numbers or symbols) which is such that the object and the difficulty of the question remain unaltered, mark according to the scheme but following through from the candidate's data. A penalty is then applied; 1 mark is generally appropriate, though this may differ for some units. This is achieved by withholding one A mark in the question.

Note that a miscopy of the candidate's own working is not a misread but an accuracy error.

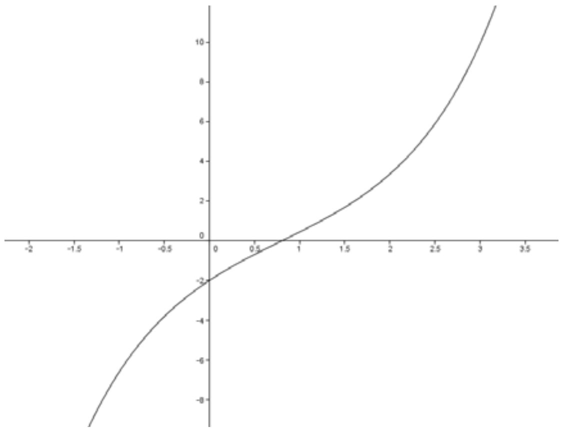
Question			Answer	Marks	Guidance	
1	(a)	(i)	$a\tan y = x \Rightarrow a\sec^2 y\dfrac{dy}{dx} = 1$	M1	Differentiating with respect to x or y	$\dfrac{dx}{dy} = a\sec^2 y$
			$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{a\sec^2 y}$	A1	For $\dfrac{dy}{dx}$	Or $a\dfrac{dy}{dx} = \dfrac{1}{\sec^2 y}$
			$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{a\left(1+\dfrac{x^2}{a^2}\right)}$			
			$\Rightarrow \dfrac{dy}{dx} = \dfrac{a}{a^2+x^2}$	A1(ag)	Completion www with sufficient detail	
				[3]		
1	(a)	(ii)	$x^2 - 4x + 8 = (x-2)^2 + 4$	B1		
			$\int_0^4 \dfrac{1}{x^2-4x+8}\,dx = \dfrac{1}{2}\left[\arctan\dfrac{x-2}{2}\right]_0^4$	M1 A1	Integral of form a arctan bu or any appropriate substitution Correct integral with consistent limits	$\dfrac{1}{2}\left[\arctan\dfrac{u}{2}\right]_{-2}^{2}$
			$= \dfrac{1}{2}\left(\arctan(1) - \arctan(-1)\right)$			
			$= \dfrac{\pi}{4}$	A1	Evaluated in terms of π	
				[4]		
1	(a)	(iii)	$\int 1\times\arctan x\,dx$	M1	Using parts with $u = \arctan x$ and $v' = 1$	Allow one other error
			$= x\arctan x - \int \dfrac{x}{1+x^2}\,dx$	A1		
				M1	$\int \dfrac{x}{1+x^2}\,dx = a\ln\left(1+x^2\right)$	
			$= x\arctan x - \dfrac{1}{2}\ln\left(1+x^2\right) + c$	A1	$a = \dfrac{1}{2}$. Condone omitted c	
				[4]		

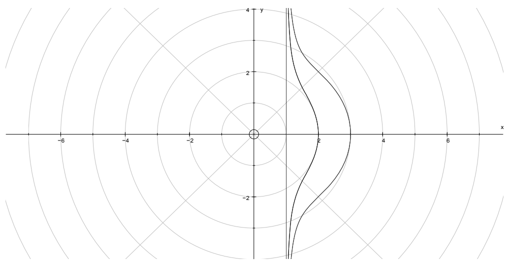
Question			Answer	Marks	Guidance	
1	(b)	(i)	$r = 2\cos\theta \Rightarrow r^2 = 2r\cos\theta$	M1	Using $r^2 = x^2 + y^2$ and $x = r\cos\theta$	
			$\Rightarrow x^2 + y^2 = 2x$	A1	A correct cartesian equation in any form	
			$\Rightarrow (x-1)^2 + y^2 = 1$	A1(ag)	Explaining that the curve is a circle	e.g. writing as $(x-\alpha)^2 + (y-\beta)^2 = r^2$
			OR $x = r\cos\theta \Rightarrow x = 2\cos^2\theta$ $y = r\sin\theta \Rightarrow y = 2\cos\theta\sin\theta = \sin 2\theta$ M1 $\cos 2\theta = 2\cos^2\theta - 1 \Rightarrow x = \cos 2\theta + 1$ A1 $\Rightarrow (x-1)^2 + y^2 = 1$ A1(ag)		Using $x = r\cos\theta$, $y = r\sin\theta$ and linking x in terms of $\cos 2\theta$ Explaining that the curve is a circle	e.g. writing as $(x-\alpha)^2 + (y-\beta)^2 = r^2$
			Centre (1, 0) Radius 1	B1 B1 **[5]**	Independent Independent	
1	(b)	(ii)	$x^2 + (y-2)^2 = 4 \Rightarrow x^2 + y^2 = 4y$ $\Rightarrow r^2 = 4r\sin\theta$ $\Rightarrow r = 4\sin\theta$	 M1 A1 **[2]**	Using $r^2 = x^2 + y^2$ and $y = r\sin\theta$	For answer alone www: B1 for $r = k\sin\theta$, B1 for $k = 4$
2	(a)	(i)	$1 + e^{j2\theta} = 1 + \cos 2\theta + j\sin 2\theta$ $= 1 + (2\cos^2\theta - 1) + 2j\sin\theta\cos\theta$ $= 2\cos^2\theta + 2j\sin\theta\cos\theta$ $= 2\cos\theta(\cos\theta + j\sin\theta)$	 M1 A1(ag)	Using $e^{2j\theta} = \cos 2\theta + j\sin 2\theta$ and double angle formulae Completion www	Allow one error
			OR $1 + e^{j2\theta} = e^{j\theta}(e^{-j\theta} + e^{j\theta})$ M1 $= (\cos\theta + j\sin\theta) \times 2\cos\theta$ A1(ag)		"Factorising" and complete replacement by trigonometric functions Completion www	
			OR $1 + e^{j2\theta} = 1 + (\cos\theta + j\sin\theta)^2$ $= 1 + \cos^2\theta - \sin^2\theta + 2j\sin\theta\cos\theta$ $= 2\cos^2\theta + 2j\sin\theta\cos\theta$ M1 $= 2\cos\theta(\cos\theta + j\sin\theta)$ A1(ag)		Using $e^{j\theta} = \cos\theta + j\sin\theta$ and $1 - \sin^2\theta = \cos^2\theta$ Completion www	
				[2]		

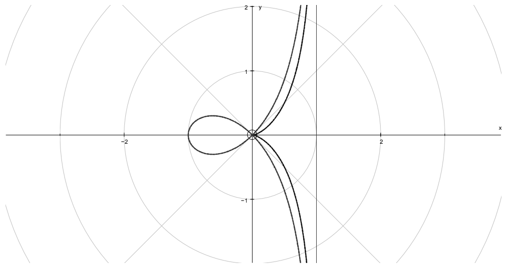
Question			Answer	Marks	Guidance	
2	(a)	(ii)	$C + jS = 1 + \binom{n}{1}e^{j2\theta} + \binom{n}{2}e^{j4\theta} + \ldots + e^{j2n\theta}$	M1	Forming $C + jS$	
			$= \left(1 + e^{j2\theta}\right)^n$	M1 A1	Recognising as binomial expansion	
			$= 2^n \cos^n \theta (\cos\theta + j\sin\theta)^n$			
			$= 2^n \cos^n \theta (\cos n\theta + j\sin n\theta)$	M1 A1	Applying (i) and De Moivre o.e.	Dependent on M1M1 above
			$\Rightarrow C = 2^n \cos^n \theta \cos n\theta$	A1(ag)	Completion www	Need to see $e^{jn\theta} = \cos n\theta + j\sin n\theta$ o.e.
			and $S = 2^n \cos^n \theta \sin n\theta$	A1 **[7]**		
2	(b)	(i)	$e^{j\frac{2\pi}{3}} = \cos\frac{2\pi}{3} + j\sin\frac{2\pi}{3} = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$	B1 **[1]**	Must evaluate trigonometric functions	
2	(b)	(ii)	Other two vertices are $(2+4j)e^{j\frac{2\pi}{3}}$	M1	Award for idea of rotation by $\frac{2\pi}{3}$	e.g. use of $\arctan 2 + \frac{2\pi}{3}$ (3.202 rad) (must be 2)
			$= (2+4j)\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$			
			$= \left(-1 - 2\sqrt{3}\right) + j\left(-2 + \sqrt{3}\right)$	A1A1	May be given as co-ordinates	
			and $(2+4j)e^{j\frac{4\pi}{3}} = (2+4j)e^{-j\frac{2\pi}{3}}$	M1	Award for idea of rotation by $-\frac{2\pi}{3}$	e.g. use of $\arctan 2 + \frac{4\pi}{3}$ (5.296 rad) (must be 2)
			$= (2+4j)\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)$			
			$= \left(-1 + 2\sqrt{3}\right) + j\left(-2 - \sqrt{3}\right)$	A1A1 **[6]**	May be given as co-ordinates	If A0A0A0A0 award SC1 for awrt −4.46 − 0.27j and 2.46 − 3.73j

Question			Answer	Marks	Guidance	
2	(b)	(iii)	Length of (2 + 4j) = $\sqrt{20}$			
			So length of side = $2\sqrt{20}\cos\frac{\pi}{6} = 2\sqrt{20}\times\frac{\sqrt{3}}{2}$	M1	Complete method	Alternative: finding distance between (2, 4) and $\left(-1-2\sqrt{3},-2+\sqrt{3}\right)$ o.e.
			$=2\sqrt{15}$	A1(ag) **[2]**	Completion www	
3	(i)		$\mathbf{M}-\lambda\mathbf{I}=\begin{pmatrix}1-\lambda & 3 & 0\\ 3 & -2-\lambda & -1\\ 0 & -1 & 1-\lambda\end{pmatrix}$			
			$\det(\mathbf{M}-\lambda\mathbf{I})$	M1	Forming $\det(\mathbf{M}-\lambda\mathbf{I})$	
			$=(1-\lambda)\left[(-2-\lambda)(1-\lambda)-1\right]-3\left[3(1-\lambda)\right]$	A1	Any correct form	Sarrus: $(1-\lambda)^2(-2-\lambda)-10(1-\lambda)$
			$=(1-\lambda)(\lambda^2+\lambda-3)-9(1-\lambda)$			or e.g. $\lambda-1+(1-\lambda)(\lambda^2+\lambda-11)$
			$\Rightarrow \lambda^3-13\lambda+12=0$	A1(ag) **[3]**	Condone omission of 0	
3	(ii)		$(\lambda-1)(\lambda^2+\lambda-12)=0$	M1	Factorising as far as quadratic	Allow one error
			$\Rightarrow(\lambda-1)(\lambda-3)(\lambda+4)=0$	A1		
			$\Rightarrow$ eigenvalues are 1, 3, −4	A1		
			$\lambda=1:\ \begin{pmatrix}0 & 3 & 0\\ 3 & -3 & -1\\ 0 & -1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$	M2	<u>For any one of λ = 1, 3, −4</u> Obtaining two independent equations	From which an eigenvector could be found
				M1	Obtaining a non-zero eigenvector	
			$\Rightarrow y=0, 3x-z=0$	A1	o.e.	Allow e.g. $3y=0, 3x-3y-z=0$
			$\Rightarrow$ eigenvector is $\begin{pmatrix}1\\ 0\\ 3\end{pmatrix}$	A1		
			$\lambda=3:\ \begin{pmatrix}-2 & 3 & 0\\ 3 & -5 & -1\\ 0 & -1 & -2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$			
			$\Rightarrow -2x+3y=0, -y-2z=0$	A1	o.e.	

Question			Answer	Marks	Guidance	
			$\Rightarrow y=-2z,\ x=-3z$ $\Rightarrow$ eigenvector is $\begin{pmatrix}-3\\-2\\1\end{pmatrix}$	A1		
			$\lambda=-4:\ \begin{pmatrix}5&3&0\\3&2&-1\\0&-1&5\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ $\Rightarrow 5x+3y=0,\ -y+5z=0$	A1	o.e.	
			$\Rightarrow y=5z,\ x=-3z$ $\Rightarrow$ eigenvector is $\begin{pmatrix}-3\\5\\1\end{pmatrix}$	A1		
				[12]		
3	**(iii)**		E.g. $\mathbf{P}=\begin{pmatrix}1&-3&-3\\0&-2&5\\3&1&1\end{pmatrix}$	B1	Use of eigenvectors (ft) as columns	
			$\mathbf{D}=\begin{pmatrix}1&0&0\\0&3^n&0\\0&0&(-4)^n\end{pmatrix}$	M1	Use of 1, 3, −4 (ft) in correct order	n not required for M1
				A1	Power n	-4^n A0
				[3]		

Question			Answer	Marks	Guidance	
4	(i)		$y = 3\sinh x - 2\cosh x$			$\frac{1}{2}e^{x} - \frac{5}{2}e^{-x}$
			$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3\cosh x - 2\sinh x$	B1		$\frac{1}{2}e^{x} + \frac{5}{2}e^{-x}$
			At TPs, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow \tanh x = \frac{3}{2}$	M1	Considering $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$	$e^{2x} = -5$; $e^{x} > 0$ and $e^{-x} > 0$
			which has no (real) solutions	A1(ag)	Showing no real roots www	
			$y = 0 \Rightarrow \tanh x = \frac{2}{3}$	M1	Solving $y = 0$ as far as e^{2x} or tanh x etc.	$e^{2x} = 5$; $\cosh x = \frac{3}{\sqrt{5}}$; $\sinh x = \frac{2}{\sqrt{5}}$
			$\Rightarrow x = \frac{1}{2}\ln\frac{1+\frac{2}{3}}{1-\frac{2}{3}}$	M1	Solving as far as x	<u>Attempt to verify</u> Award M1 for substituting $x = \frac{1}{2}\ln 5$
			$\Rightarrow x = \frac{1}{2}\ln 5$	A1(ag)	Completion www	and M1 for clearly attempting to evaluate exactly
			$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 3\sinh x - 2\cosh x = y$			
			so $y = 0 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$	B1(ag) **[7]**		$3\sinh\left(\frac{1}{2}\ln 5\right) - 2\cosh\left(\frac{1}{2}\ln 5\right) = 0$ must be explained, e.g. connected with $y = 0$
4	(ii)			B2 **[2]**	For a curve with the following features: • increasing • intersecting the positive x-axis • (0, −2) indicated • gradient increasing with large $\lvert x\rvert$ • one point of inflection	Award B1 for a curve lacking one of these features

Question		Answer	Marks	Guidance	
4	(iii)	$(3\sinh x - 2\cosh x)^2$			
		$= 9\sinh^2 x - 12\sinh x\cosh x + 4\cosh^2 x$	B1		
		$= \frac{9}{2}(\cosh 2x - 1) - 6\sinh 2x + 2(\cosh 2x + 1)$	M1	Using double "angle" formulae or complete alternative	Condone sign errors but need $\frac{1}{2}$ s
		$= \frac{13}{2}\cosh 2x - 6\sinh 2x - \frac{5}{2}$	A1	Accept unsimplified	$\frac{1}{4}e^{2x} + \frac{25}{4}e^{-2x} - \frac{5}{2}$
		$V = \pi\int_0^{\frac{1}{2}\ln 5} y^2\,\mathrm{d}x$	M1	Attempting to integrate their y^2 (ignore limits)	
		$= \pi\left[\frac{13}{4}\sinh 2x - 3\cosh 2x - \frac{5}{2}x\right]_0^{\frac{1}{2}\ln 5}$	A2	Correct results and limits c.a.o. Ignore omitted π	Give A1 for one error, or for all three terms correct and incorrect limits
		$= \pi\left[\frac{13}{4}\times\frac{12}{5} - 3\times\frac{13}{5} - \frac{5}{4}\ln 5 + 3\right]$	M1 M1	Substituting both of their limits Obtaining exact values of $\sinh(\ln 5)$ and $\cosh(\ln 5)$	$\sinh(\ln 5) = \frac{12}{5}$, $\cosh(\ln 5) = \frac{13}{5}$
		OR $= \pi\left[\frac{1}{8}e^{2x} - \frac{25}{8}e^{-2x} - \frac{5}{2}x\right]_0^{\frac{1}{2}\ln 5}$ A2		Correct results and limits	Give A1 for one error, or for all three terms correct and incorrect limits
		$= \pi\left[\frac{5}{8} - \frac{5}{8} - \frac{5}{4}\ln 5 + 3\right]$ M1 M1		Substituting both of their limits Obtaining exact values of e^{2x} and e^{-2x}	$e^{2x} = 5$, $e^{-2x} = \frac{1}{5}$
		$= \pi\left[3 - \frac{5}{4}\ln 5\right]$	A1(ag) **[9]**	Completion www	
5	(i)		B2 B1 **[3]**	Three curves of correct shape Correctly identified	Give B1 for two correct curves $a = 0$, $a = 1$, $a = 2$ from left to right

Question			Answer	Marks	Guidance	
5	(ii)			B1 B1 **[2]**	Curve for $a = -1$ Curve for $a = -2$	Curve with cusp Curve with loop
5	(iii)		Asymptote	B1 **[1]**		
5	(iv)		$a = -1$: cusp $a = -2$: loop	B1 B1 **[2]**		
5	(v)		$r = \sec\theta + a\cos\theta \Rightarrow r\cos\theta = 1 + a\cos^2\theta$ $\Rightarrow x = 1 + a\left(\frac{x^2}{r^2}\right)$ $\Rightarrow x - 1 = a\left(\frac{x^2}{x^2+y^2}\right)$ $\Rightarrow x^2 + y^2 = a\left(\frac{x^2}{x-1}\right) \Rightarrow y^2 = a\left(\frac{x^2}{x-1}\right) - x^2$ Hence asymptote at $x = 1$	M1 M1 M1 A1(ag) B1 **[5]**	Using $x = r\cos\theta$ Using $r^2 = x^2 + y^2$ Making y^2 subject	
5	(vi)		Curve exists for $y^2 \geq 0$ $\Rightarrow a\left(\frac{1}{x-1}\right) - 1 \geq 0$ If $a > 0$ then $x - 1 > 0$ and so $a \geq x - 1$ i.e. $1 < x \leq 1 + a$ If $a < 0$ then $x - 1 < 0$ and so $a \leq x - 1$ i.e. $1 + a \leq x < 1$	M1 M1 A1(ag) M1 A1 **[5]**	Considering $y^2 \geq 0$	

OCR (Oxford Cambridge and RSA Examinations)
1 Hills Road
Cambridge
CB1 2EU

OCR Customer Contact Centre

Education and Learning
Telephone: 01223 553998
Facsimile: 01223 552627
Email: general.qualifications@ocr.org.uk

www.ocr.org.uk

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