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1)
$$e^{2x} - 5e^{x} = 0$$
$$e^{x}(e^{x}-5) = 0$$
$$\Rightarrow e^{x} - 5 = 0$$
$$e^{x} = 5$$
$$\underline{x = \ln 5}$$

2) $T = 20 + be^{-kt}$

At $t=0$, $T = 100$

At $t=5$, $T = 60$

i) Subst $t=0, T=100$
$$100 = 20 + be^{0}$$
$$\Rightarrow \underline{b = 80}$$

Subst $t=5$, $T=60$
$$60 = 20 + 80e^{-5k}$$
$$40 = 80e^{-5k}$$
$$\frac{40}{80} = e^{-5k}$$
$$\ln\left(\tfrac{1}{2}\right) = -5k$$
$$k = \frac{\ln\left(\frac{1}{2}\right)}{-5}$$
$$\underline{k = 0.1386}$$

2ii) If $T = 50°c$
$$50 = 20 + 80e^{-0.1386t}$$
$$30 = 80e^{-0.1386t}$$
$$\frac{30}{80} = e^{-0.1386t}$$
$$\ln\left(\tfrac{3}{8}\right) = -0.1386t$$
$$t = \frac{\ln\left(\frac{3}{8}\right)}{-0.1386}$$
$$t = 7.075 \text{ mins}$$

3) i)
$$y = \sqrt[3]{1+3x^2}$$
$$y = (1+3x^2)^{\frac{1}{3}}$$
Let $u = 1+3x^2$
$$\frac{du}{dx} = 6x$$
$$y = u^{\frac{1}{3}}$$
$$\frac{dy}{du} = \frac{1}{3}u^{-\frac{2}{3}}$$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \frac{1}{3}u^{-2/3} \times 6x$$
$$= \frac{2x}{(1+3x^2)^{2/3}}$$

3 ii) $y^3 = 1 + 3x^2$

$$3y^2 \frac{dy}{dx} = 6x$$

$$\frac{dy}{dx} = \frac{6x}{3y^2} = \frac{2x}{y^2}$$

However, since $y = \sqrt[3]{1+3x^2}$

$$\frac{dy}{dx} = \frac{2x}{(1+3x^2)^{2/3}}$$

same result as part (i)

4) i)

$$\int_0^1 \frac{2x}{x^2+1}\,dx$$

$$= \Big[\ln(x^2+1)\Big]_0^1$$

$$= \ln 2 - \ln 1$$

$$= \ln 2$$

4 ii)

$$\int_0^1 \frac{2x}{x+1}\,dx$$

Let $u = x+1$

$\Rightarrow \frac{du}{dx} = 1$

$du = dx$

Also $x = u - 1$

Change limits $x = 1,\ u = 2$
$x = 0,\ u = 1$

$$\int_1^2 \frac{2(u-1)}{u}\,du$$

$$= 2\int_1^2 \left(1 - \frac{1}{u}\right) du$$

$$= 2\Big[u - \ln u\Big]_1^2$$

$$= 2\Big[(2 - \ln 2) - (1 - \ln 1)\Big]$$

$$= 2\Big[1 - \ln 2\Big]$$

$$= 2 - 2\ln 2$$

5) i) $y = a + b\sin cx$

$a = 0,\quad b = 3,\quad c = 2$

ii) $a = 1,\quad b = -1,\quad c = 1$

6) for $f(x)$ to be odd function

$f(-x) = -f(x)$ for all x in domain

for $g(x)$ to be an even function

$g(-x) = g(x)$ for all x in domain

Assume $f(x)$ is odd, $g(x)$ is even

then $gf(-x) = g(-f(x))$

$= g(f(x))$

$= gf(x)$

$\therefore gf(x)$ is even.

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7) $\arcsin x = \arccos y$

or $\sin^{-1}x = \cos^{-1}y$

Let $\sin^{-1}x = \theta$

then $\cos^{-1}y = \theta$ also

$\Rightarrow \quad x = \sin\theta$

and $\quad y = \cos\theta$

$\therefore \quad x^2 = \sin^2\theta$

and $\quad y^2 = \cos^2\theta$

$\therefore \quad x^2 + y^2 = \sin^2\theta + \cos^2\theta$

$x^2 + y^2 = 1$

Section B

8) $y = x\cos 3x$

i) At O, P, Q $\quad y = 0$

$\Rightarrow \quad x\cos 3x = 0$

$\Rightarrow \quad x = 0$

or $\cos 3x = 0$

$\Rightarrow \quad 3x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

$\Rightarrow \quad x = \frac{\pi}{6}$ or $\frac{\pi}{2}$

for first two intersections with x axis after O

so $P\left(\frac{\pi}{6}, 0\right)$, $Q\left(\frac{\pi}{2}, 0\right)$

ii) $y = x\cos 3x$

$$\frac{dy}{dx} = x(-3\sin 3x) + \cos 3x$$

$$\frac{dy}{dx} = \cos 3x - 3x\sin 3x$$

At P where $x = \frac{\pi}{6}$

$$\frac{dy}{dx} = \cos\left(\frac{3\pi}{6}\right) - 3 \times \frac{\pi}{6} \times \sin\left(\frac{3\pi}{6}\right)$$

$$= 0 - \frac{\pi}{2} = -\frac{\pi}{2}$$

Gradient at P is $-\frac{\pi}{2}$

At turning points $\frac{dy}{dx} = 0$

$\Rightarrow \quad \cos 3x - 3x\sin 3x = 0$

$\Rightarrow \quad \cos 3x = 3x\sin 3x$

$\Rightarrow \quad 1 = \dfrac{3x\sin 3x}{\cos 3x}$

$\Rightarrow \quad \frac{1}{3} = x\tan 3x$

iii) $$\text{Area} = \int_0^{\frac{\pi}{6}} x\cos 3x \, dx$$

Let $u = x$ $\quad\Rightarrow \frac{du}{dx} = 1$

Let $\frac{dv}{dx} = \cos 3x$ $\quad\Rightarrow v = \frac{1}{3}\sin 3x$

Using $\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$

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8 iii) cont)

$$\int_0^{\frac{\pi}{6}} x\cos 3x$$

$$= \left[x\frac{1}{3}\sin 3x\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \frac{1}{3}\sin 3x\,dx$$

$$= \left[\frac{x}{3}\sin 3x\right]_0^{\frac{\pi}{6}} - \left[-\frac{1}{9}\cos 3x\right]_0^{\frac{\pi}{6}}$$

$$= \left(\frac{\pi}{18} - 0\right) - \left(0 - -\frac{1}{9}\right)$$

$$= \frac{\pi}{18} - \frac{1}{9}$$

9)

$$f(x) = \frac{2x^2-1}{x^2+1} \qquad 0 \le x \le 2$$

i)

$$f'(x) = \frac{(x^2+1)4x - (2x^2-1)2x}{(x^2+1)^2}$$

$$f'(x) = \frac{4x^3+4x-4x^3+2x}{(x^2+1)^2}$$

$$f'(x) = \frac{6x}{(x^2+1)^2}$$

ii) for $0 \le x \le 2$

$f'(x)$ is always ≥ 0

so $f(x)$ is increasing

$\therefore$ lowest value when $x = 0$

highest value when $x = 2$

$$f(0) = \frac{0-1}{0+1} = -1$$

$$f(2) = \frac{2(2)^2-1}{2^2+1} = \frac{7}{5}$$

Range $-1 \le f(x) \le \frac{7}{5}$

iii) At max value of $f'(x)$

$f''(x) = 0$

$$\Rightarrow \frac{6-18x^2}{(x^2+1)^3} = 0$$

$$\Rightarrow 6 - 18x^2 = 0$$

$$\Rightarrow 18x^2 = 6$$

$$\Rightarrow x^2 = \frac{1}{3}$$

$$\Rightarrow x = \pm\frac{1}{\sqrt{3}}$$

$$\Rightarrow x = \frac{1}{\sqrt{3}} \quad \text{since } 0 \le x \le 2$$

Max value of $f'(x)$

$$= \frac{6\left(\frac{1}{\sqrt{3}}\right)}{\left(\left(\frac{1}{\sqrt{3}}\right)^2+1\right)^2}$$

$$= \frac{\frac{6}{\sqrt{3}}}{\left(\frac{1}{3}+1\right)^2}$$

$$= \frac{\frac{6}{\sqrt{3}}}{\frac{16}{9}}$$

$$= \frac{6}{\sqrt{3}} \times \frac{3}{16}$$

$$= \frac{6\sqrt{3}}{16} = \frac{3\sqrt{3}}{8}$$

9iii) So max value of

$$f'(x) = \frac{3\sqrt{3}}{8}$$

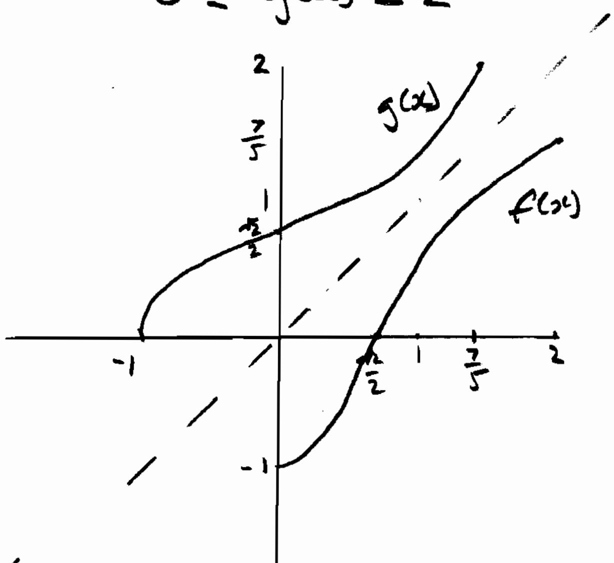
9iv)

domain of $g(x)$ is given by

$$-1 \le x \le \frac{7}{5}$$

range of $g(x)$ is given by

$$0 \le g(x) \le 2$$



($y = x$ is line of symmetry between the graphs of $f(x)$ and $g(x)$)

9v)

$$f(x) = \frac{2x^2-1}{x^2+1}$$

Let $$y = \frac{2x^2-1}{x^2+1}$$

Swap variables and rearrange

$$x = \frac{2y^2-1}{y^2+1}$$

$$x(y^2+1) = 2y^2-1$$

$$xy^2+x = 2y^2-1$$

$$x+1 = 2y^2-xy^2$$

$$x+1 = y^2(2-x)$$

$$\frac{x+1}{2-x} = y^2$$

$$\therefore\ y = \sqrt{\frac{x+1}{2-x}}$$

$$\therefore\ f^{-1}(x) = g(x) = \sqrt{\frac{x+1}{2-x}}$$