MEI CORE 3 JANUARY 2013

1) i)

$$y = e^{-x}\sin 2x$$

$$\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = e^{-x} \times 2\cos 2x - e^{-x}\sin 2x$$

$$\frac{dy}{dx} = e^{-x}(2\cos 2x - \sin 2x)$$

ii) For st.pt. $\frac{dy}{dx} = 0$

$$\Rightarrow 2\cos 2x - \sin 2x = 0$$

$$\Rightarrow 2\cos 2x = \sin 2x$$

$$\Rightarrow 2 = \frac{\sin 2x}{\cos 2x} = \tan 2x$$

$$\Rightarrow 2x = \tan^{-1} 2$$

$$\Rightarrow x = \frac{1}{2}\tan^{-1} 2$$

2) i)

$$x^2 + 2y^2 = 4x$$

$$2x + 4y\frac{dy}{dx} = 4$$

$$4y\frac{dy}{dx} = 4 - 2x$$

$$\frac{dy}{dx} = \frac{4-2x}{4y}$$

$$\frac{dy}{dx} = \frac{2-x}{2y}$$

ii) At st. pt. $\frac{dy}{dx} = 0$

$$\Rightarrow 2 - x = 0$$

$$\Rightarrow x = 2$$

Sub for x

$$2^2 + 2y^2 = 4(2)$$

$$2y^2 = 8 - 4$$

$$2y^2 = 4$$

$$y^2 = 2$$

$$y = \pm\sqrt{2}$$

stationary points

$(2, \sqrt{2})$ and $(2, -\sqrt{2})$

3) $1 < x < 3$

Midpoint of interval is 2
Distance from midpoint to extremes is 1

$$|x - 2| < 1$$

ie distance of x from 2 is less than 1

4) $\theta = a - be^{-kt}$

When $t = 0$, $\theta = 15$

when $t = \infty$, $\theta = 100$

MEI CORE3 JANUARY 2013

4 i) Subst $t=0,\ \theta=15$

$$15 = a - b \quad (1)$$

subst $t=\infty,\ \theta=100$

$$100 = a \quad (2)$$

Sub for a in (1)

$$15 = 100 - b$$

$$\Rightarrow b = 85$$

$\therefore\ a = 100,\ b = 85$

Sub $t=1,\ \theta=30$

$$30 = 100 - 85e^{-k}$$

$$85e^{-k} = 100 - 30$$

$$e^{-k} = \frac{70}{85}$$

$$-k = \ln\left(\frac{70}{85}\right)$$

$$k = 0.194 \quad \text{to 3 s.f.}$$

4ii) $$\theta = 100 - 85e^{-0.194t}$$

when $\theta = 80$

$$80 = 100 - 85e^{-0.194t}$$

$$85e^{-0.194t} = 100 - 80$$

$$e^{-0.194t} = \frac{20}{85}$$

$$-0.194t = \ln\left(\frac{20}{85}\right)$$

$$t = \frac{\ln\left(\frac{20}{85}\right)}{-0.194}$$

$$t = 7.46 \text{ mins} \quad \text{to 3 s.f.}$$

5) i) $$F = \frac{25}{v} = 25v^{-1}$$

$$\frac{dF}{dv} = -25v^{-2} = -\frac{25}{v^2}$$

ii) $$\frac{dF}{dt} = \frac{dF}{dv} \times \frac{dv}{dt}$$

when $v = 50$ and $\frac{dv}{dt} = 1.5$

$$\frac{dF}{dt} = -\frac{25}{v^2} \times \frac{dv}{dt}$$

$$\frac{dF}{dt} = \frac{-25}{50^2} \times 1.5$$

$$\frac{dF}{dt} = -0.015 \quad \text{N s}^{-1}$$

6) $$\int_0^3 x(x+1)^{-\frac{1}{2}}\,dx$$

Let $u = x+1$

$$\frac{du}{dx} = 1$$

$$du = dx$$

MEI CORE 3 JANUARY 2013

6 cont) Also $x = u - 1$

when $x = 0$, $u = 1$
when $x = 3$, $u = 4$

$$\therefore \int_0^3 x(x+1)^{-\frac{1}{2}}\,dx$$

$$= \int_1^4 (u-1)u^{-\frac{1}{2}}\,du$$

$$= \int_1^4 \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right)du$$

$$= \left[\frac{u^{3/2}}{3/2} - \frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^4$$

$$= \left[\frac{2}{3}u^{3/2} - 2u^{\frac{1}{2}}\right]_1^4$$

$$= \left(\frac{2}{3}\times 8 - 2\times 2\right) - \left(\frac{2}{3} - 2\right)$$

$$= \frac{16}{3} - 4 - \frac{2}{3} + 2$$

$$= \frac{8}{3} \quad \text{or} \quad 2\tfrac{2}{3}$$

7) i) $3^n + 2$ prime for $n \geqslant 0$?

$3^1 + 2 = 5$ ✓
$3^2 + 2 = 11$ ✓
$3^3 + 2 = 29$ ✓
$3^4 + 2 = 83$ ✓
$3^5 + 2 = 245$ not prime

$3^n + 2$ is not prime for all $n \geqslant 0$

since $3^5 + 2 = 245$

which is divisible by 5

7ii)

$3^0 = 1$
$3^1 = 3$
$3^2 = 9$
$3^3 = 27$
$3^4 = 81$

The final digit when multiplying by 3 is determined by the final digit of the preceding power of 3. $\therefore$ the cycle 1, 3, 9, 7 is repeated ad infinitum

$\therefore$ no power of 3 ends in 5

Section B

8) $f(x) = \tan x$

$g(x) = 1 + f\left(x - \frac{\pi}{4}\right)$

i) Translation by $\begin{pmatrix} \frac{\pi}{4} \\ 0 \end{pmatrix}$

Translation by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

in either order

MEI CORE 3 JANUARY 2013

8 ii)

$$g(x) = \frac{2\sin x}{\sin x + \cos x}$$

$$g'(x) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$g'(x) =$$

$$\frac{(\sin x + \cos x)2\cos x - 2\sin x(\cos x - \sin x)}{(\sin x + \cos x)^2}$$

$$= \frac{\cancel{2\sin x\cos x} + 2\cos^2 x - \cancel{2\sin x\cos x} + 2\sin^2 x}{(\sin x + \cos x)^2}$$

$$= \frac{2(\cos^2 x + \sin^2 x)}{(\sin x + \cos x)^2}$$

$$= \frac{2}{(\sin x + \cos x)^2}$$

At $\left(\frac{\pi}{4}, 1\right)$

$$\text{gradient} = \frac{2}{\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right)^2}$$

$$= \frac{2}{\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)^2}$$

$$= \frac{2}{\left(\frac{2}{\sqrt{2}}\right)^2}$$

$$= \frac{2}{(\sqrt{2})^2} = \frac{2}{2} = 1$$

At origin $(0,0)$

$$f(x) = \tan x$$

$$f'(x) = \sec^2 x = \frac{1}{\cos^2 x}$$

$$f'(0) = \frac{1}{\cos^2 0} = \frac{1}{1^2} = 1$$

$\therefore$ gradient of $g(x)$ at $\left(\frac{\pi}{4}, 1\right)$ is same as gradient of $f(x)$ at $(0,0)$

8 iii)

$$\int_0^{\frac{\pi}{4}} f(x)\,dx = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\,dx$$

Let $u = \cos x$

$$\frac{du}{dx} = -\sin x$$

$$du = -\sin x\,dx$$

$$-du = \sin x\,dx$$

when $x = 0$, $u = \cos 0 = 1$

when $x = \frac{\pi}{4}$, $u = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$\therefore \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\,dx = \int_1^{\frac{1}{\sqrt{2}}} -\frac{1}{u}\,du$$

$$= \int_{\frac{1}{\sqrt{2}}}^{1} \frac{1}{u}\,du$$

$$= \left[\ln u\right]_{\frac{1}{\sqrt{2}}}^{1} = \ln 1 - \ln\frac{1}{\sqrt{2}}$$

$$= \ln(\sqrt{2})$$

MEI CORE 3 JANUARY 2013

8iv) Area required is area under $f(x)$ just found + rectangle $\frac{\pi}{4}$ by 1

$$= \ln 2 + \frac{\pi}{4}$$

This is because curve has been translated by $\begin{pmatrix} \frac{\pi}{4} \\ 1 \end{pmatrix}$

9) i)
$$y = f(x) = \tfrac{1}{2}\left(e^x - 1\right)$$

and $y = x$ intersect at (a, a)

$$\Rightarrow \tfrac{1}{2}\left(e^a - 1\right) = a$$

$$\Rightarrow e^a - 1 = 2a$$

$$\Rightarrow e^a = 2a + 1$$

ii) Area under curve between $x = 0$ and $x = a$

$$= \int_0^a \tfrac{1}{2}\left(e^x - 1\right) dx$$

$$= \left[\frac{e^x}{2} - \frac{x}{2}\right]_0^a$$

$$= \frac{e^a}{2} - \frac{a}{2} - \left(\frac{1}{2} - 0\right)$$

from (i)

$$= \frac{2a+1}{2} - \frac{a}{2} - \frac{1}{2}$$

$$= a + \frac{1}{2} - \frac{a}{2} - \frac{1}{2} = \frac{a}{2}$$

Area between curve and $y = x$

= Area under $y = x$ − area under curve

$$= \frac{a^2}{2} - \frac{a}{2}$$

9iii) Let $y = \tfrac{1}{2}\left(e^x - 1\right)$

Swap variables and rearrange

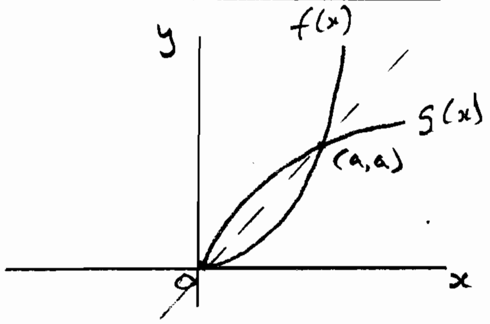
$$x = \tfrac{1}{2}\left(e^y - 1\right)$$

$$2x = e^y - 1$$

$$2x + 1 = e^y$$

$$\ln(2x+1) = y$$

$$\therefore f^{-1}(x) = \ln(2x+1) = g(x)$$



9iv)
$$f(x) = \tfrac{1}{2}\left(e^x - 1\right)$$

$$f'(x) = \tfrac{1}{2}e^x$$

$$g(x) = \ln(2x+1)$$

$$g'(x) = \frac{2}{2x+1}$$

$$f'(a) = \frac{e^a}{2} = \frac{2a+1}{2}$$

$$g'(a) = \frac{2}{2a+1} \qquad \therefore g'(a) = \frac{1}{f'(a)}$$

$f(x)$ and $g(x)$ are reflections of each other in $y = x$