MEI CORE 3 JUNE 2009

1)
$$\int_0^{\frac{\pi}{6}} \sin 3x \, dx$$
$$= \left[-\frac{1}{3}\cos 3x\right]_0^{\frac{\pi}{6}}$$
$$= \left(-\frac{1}{3}\cos\frac{\pi}{2}\right) - \left(-\frac{1}{3}\cos 0\right)$$
$$= 0 - -\frac{1}{3} = \frac{1}{3}$$

2) $M = Ae^{-kt}$

i) Given $M = 100$ when $t = 0$

$100 = Ae^{0} = A$

$\Rightarrow A = 100$

Given $M = 50$ when $t = 1500$

$50 = 100e^{-1500k}$

$\frac{50}{100} = e^{-1500k}$

$\ln\left(\frac{1}{2}\right) = -1500k$

$k = \dfrac{\ln\left(\frac{1}{2}\right)}{-1500}$

$k = 4.62 \times 10^{-4}$

ii) Safe when only 1g of original 100g is left

$1 = 100e^{-0.000462t}$

$\frac{1}{100} = e^{-0.000462t}$

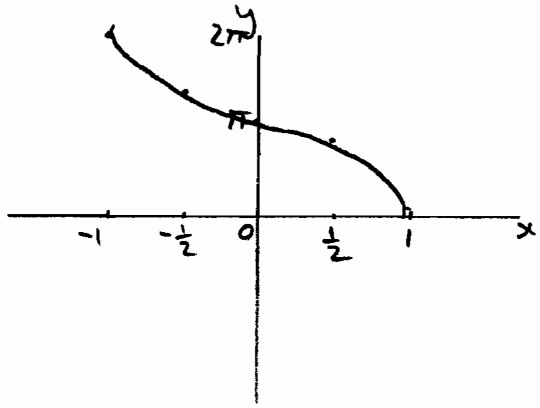
$\ln\left(\frac{1}{100}\right) = -0.000462t$

$t = \dfrac{\ln\left(\frac{1}{100}\right)}{-0.000462}$

$t = 9968$ years

$t = 9970$ years to 3 s.f

3) $y = 2\arccos x \quad -1 \le x \le 1$



$2\cos^{-1}(-1) = 2\pi$

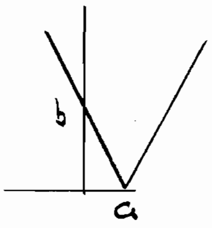
$2\cos^{-1}(-\frac{1}{2}) = \frac{4\pi}{3}$

$2\cos^{-1}(0) = \pi$

$2\cos^{-1}(\frac{1}{2}) = \frac{2\pi}{3}$

$2\cos^{-1}(1) = 0$

4) $y = 2|x-1|$



$a = 1$

when $x = 0$ $\quad y = 2|-1|$

$y = 2$

$\therefore b = 2$

5)
$$e^{2y} = 1 + \sin x$$

i)
$$2e^{2y}\frac{dy}{dx} = \cos x$$

$$\frac{dy}{dx} = \frac{\cos x}{2e^{2y}}$$

ii)
$$2y = \ln(1+\sin x)$$

$$y = \tfrac{1}{2}\ln(1+\sin x)$$

$$\frac{dy}{dx} = \frac{1}{2} \times \frac{\cos x}{1+\sin x}$$

$$\frac{dy}{dx} = \frac{\cos x}{2e^{2y}}$$

6)
$$f(x) = \frac{x+1}{x-1}$$

$$ff(x) = f\left(\frac{x+1}{x-1}\right)$$

$$= \frac{\left(\frac{x+1}{x-1}\right)+1}{\left(\frac{x+1}{x-1}\right)-1}$$

Multiply top and bottom by $x-1$

$$= \frac{(x+1)+(x-1)}{(x+1)-(x-1)}$$

$$= \frac{2x}{2} = x$$

$$f^{-1}(x) = f(x) = \frac{x+1}{x-1}$$

f is its own inverse and so is symmetrical about the line $y=x$

7) A)

i)
$$(x-y)(x^2+xy+y^2)$$

$$= x^3 + \cancel{x^2y} + \cancel{xy^2} - \cancel{x^2y} - \cancel{xy^2} - y^3$$

$$= x^3 - y^3$$

B)
$$(x+\tfrac{1}{2}y)^2 + \tfrac{3}{4}y^2$$

$$= x^2 + \tfrac{1}{2}xy + \tfrac{1}{2}xy + \tfrac{1}{4}y^2 + \tfrac{3}{4}y^2$$

$$= x^2 + xy + y^2$$

ii) From B

$(x+\tfrac{1}{2}y)^2 \geqslant 0$, $\tfrac{3}{4}y^2 \geqslant 0$

At least one of these > 0

so $(x+\tfrac{1}{2}y)^2 + \tfrac{3}{4}y^2 > 0$

$\Rightarrow x^2 + xy + y^2 > 0$ for all real x, y

If $x > y$ then $(x-y) > 0$

$\Rightarrow (x-y)(x^2+xy+y^2) > 0$

since both factors are positive

from A this $\Rightarrow x^3 - y^3 > 0$

Section B

8) $f(x) = e^{x-1}$ $g(x) = 1 + \ln x$

i) At A $g(x) = 0$

$$1 + \ln x = 0$$
$$\ln x = -1$$
$$x = e^{-1}$$

A is point $(e^{-1}, 0)$

At B $x = 0$, $y = f(0)$

$$y = e^{0-1} = e^{-1}$$

B is point $(0, e^{-1})$

ii) Let $y = f(x) = e^{x-1}$

Swap variables

$$x = e^{y-1}$$
$$\ln x = y - 1$$
$$1 + \ln x = y$$
$$\therefore f^{-1}(x) = 1 + \ln x = g(x)$$

iii) $\int_0^1 f(x)\,dx$

$$= \int_0^1 e^{x-1}\,dx$$
$$= \left[e^{x-1}\right]_0^1 = e^0 - e^{-1}$$
$$= 1 - e^{-1}$$

omission from part (i)

Intersect at C

$$e^{x-1} = 1 + \ln(x)$$

when $x = 1$, $e^{x-1} = e^0 = 1$

when $x = 1$, $1 + \ln(x) = 1 + 0 = 1$

$\therefore (1,1)$ on both curves (C)

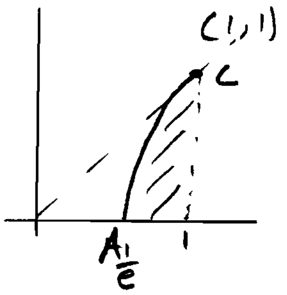
iv) $\int \ln x\,dx$ Let $u = \ln x$ $\Rightarrow \frac{du}{dx} = 1$ Let $\frac{dv}{dx} = 1$ $\Rightarrow v = x$

Using $\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$

$$\int \ln x\,dx = x\ln x - \int 1\,dx$$
$$= x\ln x - x + c$$

$$\int_{e^{-1}}^1 g(x)\,dx = \int_{e^{-1}}^1 (1 + \ln x)\,dx$$
$$= \left[x + x\ln x - x\right]_{e^{-1}}^1$$
$$= \left[x\ln x\right]_{e^{-1}}^1$$
$$= 1\ln 1 - e^{-1}\ln(e^{-1})$$
$$= 0 + e^{-1} = e^{-1} = \frac{1}{e}$$

v)



From part iv shaded area =

$$\int_{e^{-1}}^{1} g(x)\,dx = \frac{1}{e}$$

Area required =

$$2 \times \left(\underset{\text{i.e. } \triangle}{(\text{Area under } y = x)} - \text{shaded area} \right)$$

$$= 2 \times \left[\frac{1}{2} - \frac{1}{e} \right]$$

$$= 1 - \frac{2}{e}$$

9) $y = \dfrac{x^2}{3x-1}$

i) $a = \dfrac{1}{3}$

ii)
$$\frac{dy}{dx} = \frac{(3x-1)(2x) - x^2(3)}{(3x-1)^2}$$

$$= \frac{6x^2 - 2x - 3x^2}{(3x-1)^2}$$

$$= \frac{3x^2 - 2x}{(3x-1)^2}$$

$$= \frac{x(3x-2)}{(3x-1)^2}$$

iii) At st pt P $\dfrac{dy}{dx} = 0$

$$\Rightarrow x(3x-2) = 0$$

$$\Rightarrow x = 0 \text{ or } x = \frac{2}{3}$$

From graph $x = \frac{2}{3}$ at P

$$y = \frac{\left(\frac{2}{3}\right)^2}{3\left(\frac{2}{3}\right) - 1} = \frac{\frac{4}{9}}{1} = \frac{4}{9}$$

P is point $\left(\frac{2}{3}, \frac{4}{9} \right)$

When $x = 0.6$ $\dfrac{dy}{dx} = \dfrac{0.6(1.8-2)}{(1.8-1)^2}$

$$= -0.1875$$

When $x = 0.8$ $\dfrac{dy}{dx} = \dfrac{0.8(2.4-2)}{(2.4-1)^2}$

$$= 0.1633$$

$\dfrac{dy}{dx}$ $-$ $\rightarrow$ 0 $\rightarrow$ $+$

_/ $\therefore$ P a min

iv) $\displaystyle\int \frac{x^2}{3x-1}\,dx$

Let $u = 3x - 1$

$\Rightarrow \dfrac{du}{dx} = 3$

$du = 3dx$

$\frac{1}{3}du = dx$

Also $x = \dfrac{u+1}{3}$

Integral becomes

$$\int \frac{\left(\frac{u+1}{3}\right)^2}{u} \cdot \frac{1}{3}\,du$$

$$= \int \frac{(u+1)^2}{27u}\,du$$

$$= \frac{1}{27} \int \frac{u^2 + 2u + 1}{u}\,du$$

9iv cont) $= \frac{1}{27}\int\left(u + 2 + \frac{1}{u}\right) du$

When $x = \frac{2}{3}$, $u = 3(\frac{2}{3}) - 1$
$u = 1$

when $x = 1$ $u = 3(1) - 1$
$u = 2$

Area is given by

$$\frac{1}{27}\int_1^2 \left(u + 2 + \frac{1}{u}\right) du$$

$$= \frac{1}{27}\left[\frac{u^2}{2} + 2u + \ln u\right]_1^2$$

$$= \frac{1}{27}\left[\left(\frac{4}{2} + 4 + \ln 2\right) - \left(\frac{1}{2} + 2 + \ln 1\right)\right]$$

$$= \frac{1}{27}\left[6 + \ln 2 - \frac{5}{2}\right]$$

$$= \frac{1}{27}\left[\frac{7}{2} + \ln 2\right]$$

$$= \frac{7 + 2\ln 2}{54}$$

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