MEI CORE 3 — MAY 2005

1) $|3x+2| = 1$

Either
$$3x+2 = 1$$
$$3x = 1-2$$
$$3x = -1$$
$$x = -\frac{1}{3}$$

or
$$3x+2 = -1$$
$$3x = -1-2$$
$$3x = -3$$
$$x = -1$$

Solution $x = -\frac{1}{3}$ or $x = -1$

2)
$$\text{Arcsin } x = \frac{\pi}{6}$$
$$\Rightarrow x = 0.5$$
$$\text{Arccos } 0.5 = \frac{\pi}{3}$$

3)
$$f(x) = \ln x, \quad g(x) = x^3$$
for $x > 0$
$$fg(x) = f(x^3) = \ln(x^3)$$
$$= 3\ln x$$

$y = f(x)$ is mapped onto $y = fg(x)$ by a stretch scale factor 3 parallel to the y-axis.

4) $T = 30 + 20e^{-0.05t}$
$t \geq 0$

when $t = 0$, $T = 30 + 20e^0$
$$T = 30 + 20$$
$$\underline{T = 50°C}$$

$$\frac{dT}{dt} = -0.05 \times 20e^{-0.05t}$$
$$= -e^{-0.05t}$$

when $t = 0$ $\frac{dT}{dt} = -e^0 = -1$

$$\underline{\frac{dT}{dt} = -1 \text{ °C/min}}$$

When $T = 40°$
$$40 = 30 + 20e^{-0.05t}$$
$$10 = 20e^{-0.05t}$$
$$\frac{1}{2} = e^{-0.05t}$$
$$\ln\left(\frac{1}{2}\right) = -0.05t$$
$$t = \frac{\ln\left(\frac{1}{2}\right)}{-0.05}$$
$$t = 13.86 \text{ min}$$

5)
$$\int_0^1 \frac{x}{2x+1}\,dx$$

Let $u = 2x+1$
$$\Rightarrow \frac{du}{dx} = 2$$
$$du = 2dx$$

5 cont)

$$\frac{1}{2}du = dx$$

Also since

$$u = 2x+1$$
$$u-1 = 2x$$
$$\frac{u-1}{2} = x$$

When $x=1$, $u = 2\times1+1 = 3$
when $x=0$, $u = 2\times0+1 = 1$

$$\therefore \int_0^1 \frac{x}{2x+1}\,dx = \int_1^3 \frac{1}{u}\cdot\frac{(u-1)}{2}\cdot\frac{1}{2}\,du$$

$$= \frac{1}{4}\int_1^3 \frac{u-1}{u}\,du$$

$$= \frac{1}{4}\int_1^3 \left(1-\frac{1}{u}\right)du$$

$$= \frac{1}{4}\Big[u - \ln u\Big]_1^3$$

$$= \frac{1}{4}\Big[(3-\ln 3)-(1-\cancel{\ln 1})\Big]$$

$$= \frac{1}{4}\Big[2-\ln 3\Big]$$

6)

$$y = \frac{x}{2+3\ln x}$$

Usind $\frac{d}{dx}\frac{u}{v} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{(2+3\ln x)1 - x\left(\frac{3}{x}\right)}{(2+3\ln x)^2}$$

$$\frac{dy}{dx} = \frac{2+3\ln x - 3}{(2+3\ln x)^2}$$

$$\frac{dy}{dx} = \frac{3\ln x - 1}{(2+3\ln x)^2}$$

At st. pt $\frac{dy}{dx} = 0$

$$\Rightarrow 3\ln x - 1 = 0$$
$$3\ln x = 1$$
$$\ln x = \frac{1}{3}$$
$$\Rightarrow x = e^{\frac{1}{3}}$$

When $x = e^{\frac{1}{3}}$

$$y = \frac{e^{\frac{1}{3}}}{2+3\ln e^{\frac{1}{3}}}$$
$$y = \frac{e^{\frac{1}{3}}}{2+3\times\frac{1}{3}\ln e}$$
$$y = \frac{e^{\frac{1}{3}}}{2+1}$$
$$y = \frac{1}{3}e^{\frac{1}{3}}$$

St pt is $\left(e^{\frac{1}{3}}, \frac{1}{3}e^{\frac{1}{3}}\right)$

7)

$$y^2 + y = x^3 + 2x$$

Find intersections with $x=2$

$$y^2+y = 2^3+4$$
$$y^2+y = 12$$

7 cont)

$$y^2 + y - 12 = 0$$
$$(y+4)(y-3) = 0$$
$$\Rightarrow \quad y = -4 \text{ or } y = +3$$

Curve and line $x = 2$ intersect at

$$(2, 3) \text{ and } (2, -4)$$

$$y^2 + y = x^3 + 2x$$
$$2y\frac{dy}{dx} + 1\frac{dy}{dx} = 3x^2 + 2$$
$$\frac{dy}{dx}(2y+1) = 3x^2 + 2$$
$$\frac{dy}{dx} = \frac{3x^2+2}{2y+1}$$

At $(2,3)$ gradient

$$\frac{dy}{dx} = \frac{3\times 2^2 + 2}{2\times 3 + 1} = \frac{12+2}{6+1}$$
$$= \frac{14}{7} = 2$$

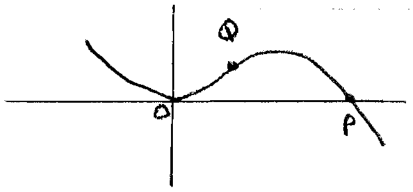
At $(2,-4)$ gradient

$$\frac{dy}{dx} = \frac{3\times 2^2 + 2}{2\times(-4) + 1} = \frac{12+2}{-8+1}$$
$$= \frac{14}{-7} = -2$$

SECTION B

8) $y = x \sin 3x$

i)



P is on x-axis

$\therefore$ at P $\quad 0 = x \sin 3x$

$$\Rightarrow x = 0 \text{ or } \sin 3x = 0$$
$$3x = \pi$$
$$x = \frac{\pi}{3}$$

x coord of P $= \frac{\pi}{3}$

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ii) At Q, $\quad x = \frac{\pi}{6}$

$$\therefore \quad y = \frac{\pi}{6}\sin\frac{3\pi}{6}$$
$$y = \frac{\pi}{6}\sin\frac{\pi}{2}$$
$$y = \frac{\pi}{6}$$

$\therefore$ Q is point $\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$

and lies on $y = x$

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iii)

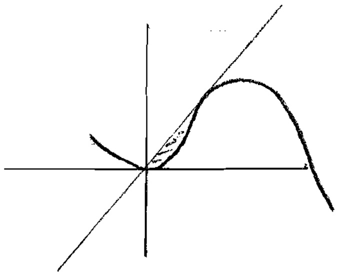
$$\frac{d}{dx} x \sin 3x$$
$$= x \times 3\cos 3x + \sin 3x \times 1$$
$$= 3x\cos 3x + \sin 3x$$

Gradient of curve at $\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$

$$= 3\times\frac{\pi}{6}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}$$
$$= 0 + 1 = 1$$

Since both curve and $y=x$ pass through $\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$ with gradient $=1$, the line $y=x$ is a tangent to the curve at that point.

iv)



Area between curve and $y=x$ given by $\int_0^{\frac{\pi}{6}} (x - x\sin 3x)\,dx$

$$= \int_0^{\frac{\pi}{6}} x(1-\sin 3x)\,dx$$

Let $u = x$ $\Rightarrow \frac{du}{dx} = 1$

Let $\frac{dv}{dx} = 1 - \sin 3x$ $\Rightarrow v = x + \frac{1}{3}\cos 3x$

Using $\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$

$$\int_0^{\frac{\pi}{6}} x(1-\sin 3x)\,dx$$

$$= \left[x\left(x+\tfrac{1}{3}\cos 3x\right)\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}}\left(x+\tfrac{1}{3}\cos 3x\right)dx$$

$$= \left[x^2 + \frac{x}{3}\cos 3x\right]_0^{\frac{\pi}{6}} - \left[\frac{x^2}{2} + \frac{1}{9}\sin 3x\right]_0^{\frac{\pi}{6}}$$

$$= \left[\left(\frac{\pi^2}{36}+0\right) - (0+0)\right] - \left[\left(\frac{\pi^2}{72} + \frac{1}{9}\sin\frac{\pi}{2}\right) - 0\right]$$

$$= \frac{\pi^2}{36} - \frac{\pi^2}{72} - \frac{1}{9}$$

$$= \frac{\pi^2}{72} - \frac{1}{9}$$

$$= \frac{\pi^2}{72} - \frac{8}{72}$$

$$= \frac{1}{72}\left(\pi^2 - 8\right)$$

9) $f(x) = \ln(1+x^2)$ for $-3 \le x \le 3$

i) $f(-x) = \ln(1+(-x)^2)$

$= \ln(1+x^2)$

$= f(x)$

$\therefore f(x)$ is an even function

The graph is therefore symmetrical about the y-axis

ii) $\frac{d}{dx}\ln(1+x^2)$

$$= \frac{2x}{1+x^2}$$

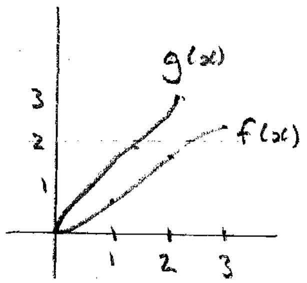
Gradient at $(2, \ln 5) = \frac{4}{1+4} = \frac{4}{5}$

9 cont) $f(x)$ has no inverse because it is not one to one based on the domain $-3 \le x \le 3$.
For example

$$f(-2) = f(2) = \ln 5$$

9 iv)

Domain now $0 \le x \le 3$



Domain of $g(x)$ is

$$0 \le x \le \ln 10$$

$f(x) = \ln(1+x^2)$
Let $y = \ln(1+x^2)$

Swap variables and change subject

$$x = \ln(1+y^2)$$

$$e^x = 1+y^2$$

$$y^2 = e^x - 1$$

$$y = \sqrt{e^x-1}$$

$$\therefore f^{-1}(x) = g(x) = \sqrt{e^x-1}$$

9 v)

$$g(x) = (e^x-1)^{\frac{1}{2}}$$

$$\frac{d}{dx}\, g(x) = \frac{1}{2}(e^x-1)^{-\frac{1}{2}} \times e^x$$

$$= \frac{e^x}{2\sqrt{e^x-1}}$$

$$g'(x) = \frac{e^x}{2\sqrt{e^x-1}}$$

When $x = \ln 5$

$$g'(\ln 5) = \frac{e^{\ln 5}}{2\sqrt{e^{\ln 5}-1}}$$

$$= \frac{5}{2\sqrt{5-1}}$$

$$= \frac{5}{2\times 2} = \frac{5}{4}$$

$$= 1\tfrac{1}{4}$$

The gradient of $g(x)$ at $(\ln 5, 2)$

$$= \frac{1}{\text{gradient of } f(x)} \text{ at } (2, \ln 5)$$

since they are corresponding points on the respective graphs