RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 4753/1

Methods for Advanced Mathematics (C3)

Wednesday **25 MAY 2005** Afternoon 1 hour 30 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF2)

TIME 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 72.

This question paper consists of 4 printed pages.

Section A (36 marks)

1 Solve the equation $|3x+2| = 1$. [3]

2 Given that $\arcsin x = \frac{1}{6}\pi$, find x. Find $\arccos x$ in terms of π. [3]

3 The functions $\mathrm{f}(x)$ and $\mathrm{g}(x)$ are defined for the domain $x > 0$ as follows:

$$\mathrm{f}(x) = \ln x, \quad \mathrm{g}(x) = x^3.$$

Express the composite function $\mathrm{fg}(x)$ in terms of $\ln x$.

State the transformation which maps the curve $y = \mathrm{f}(x)$ onto the curve $y = \mathrm{fg}(x)$. [3]

4 The temperature T °C of a liquid at time t minutes is given by the equation

$$T = 30 + 20\mathrm{e}^{-0.05t}, \quad \text{for } t \geqslant 0.$$

Write down the initial temperature of the liquid, and find the initial rate of change of temperature.

Find the time at which the temperature is 40 °C. [6]

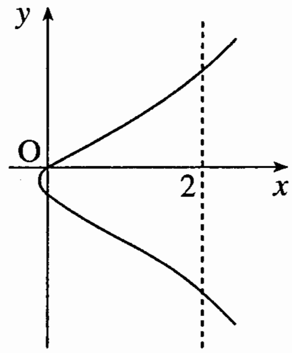
5 Using the substitution $u = 2x+1$, show that $\int_0^1 \frac{x}{2x+1}\,\mathrm{d}x = \frac{1}{4}(2-\ln 3)$. [6]

6 A curve has equation $y = \frac{x}{2+3\ln x}$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. Hence find the exact coordinates of the stationary point of the curve. [7]

7 Fig. 7 shows the curve defined implicitly by the equation

$$y^2 + y = x^3 + 2x,$$

together with the line $x = 2$.



Not to scale

Fig. 7

Find the coordinates of the points of intersection of the line and the curve.

Find $\frac{dy}{dx}$ in terms of x and y. Hence find the gradient of the curve at each of these two points. [8]

Section B (36 marks)

8 Fig. 8 shows part of the curve $y = x \sin 3x$. It crosses the x-axis at P. The point on the curve with x-coordinate $\frac{1}{6}\pi$ is Q.

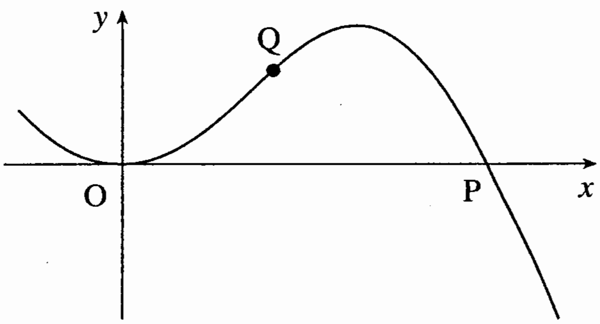


Fig. 8

(i) Find the x-coordinate of P. [3]

(ii) Show that Q lies on the line $y = x$. [1]

(iii) Differentiate $x \sin 3x$. Hence prove that the line $y = x$ touches the curve at Q. [6]

(iv) Show that the area of the region bounded by the curve and the line $y = x$ is $\frac{1}{72}(\pi^2 - 8)$. [7]

9 The function $f(x) = \ln(1 + x^2)$ has domain $-3 \leqslant x \leqslant 3$.

Fig. 9 shows the graph of $y = f(x)$.

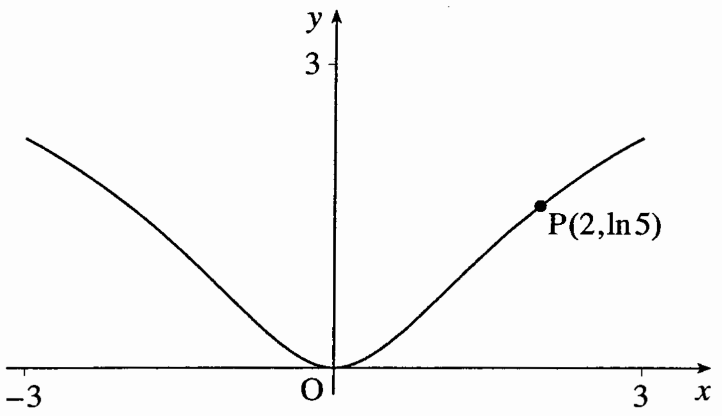


Fig. 9

(i) Show algebraically that the function is even. State how this property relates to the shape of the curve. [3]

(ii) Find the gradient of the curve at the point $P(2, \ln 5)$. [4]

(iii) Explain why the function does not have an inverse for the domain $-3 \leqslant x \leqslant 3$. [1]

The domain of $f(x)$ is now restricted to $0 \leqslant x \leqslant 3$. The inverse of $f(x)$ is the function $g(x)$.

(iv) Sketch the curves $y = f(x)$ and $y = g(x)$ on the same axes.

State the domain of the function $g(x)$.

Show that $g(x) = \sqrt{e^x - 1}$. [6]

(v) Differentiate $g(x)$. Hence verify that $g'(\ln 5) = 1\frac{1}{4}$. Explain the connection between this result and your answer to part **(ii)**. [5]

Mark Scheme 4753
June 2005

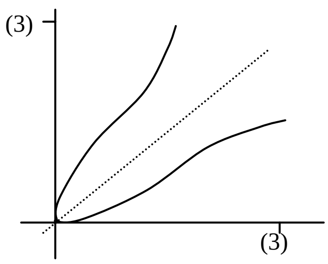
Section A

1 $3x + 2 = 1 \Rightarrow x = -1/3$ $3x + 2 = -1$ $\Rightarrow x = -1$	B1 M1 A1	$x = -1/3$ from a correct method – must be exact
or $(3x + 2)^2 = 1$ $\Rightarrow 9x^2 + 12x + 3 = 0$ $\Rightarrow 3x^2 + 4x + 1 = 0$ $\Rightarrow (3x + 1)(x + 1) = 0$ $\Rightarrow x = -1/3$ or $x = -1$	M1 B1 A1 [3]	Squaring and expanding correctly $x = -1/3$ $x = -1$
2 $x = ½$ $\cos\theta = ½$ $\Rightarrow \theta = \pi/3$	B1 M1 A1 [3]	 M1A0 for 1.04… or 60°
3 $\text{fg}(x) = \ln(x^3)$ $= 3\ln x$ Stretch s.f. 3 in y direction	M1 A1 B1 [3]	$\ln(x^3)$ $= 3\ln x$
4 $T = 30 + 20e^0 = 50$ $dT/dt = -0.05 \times 20e^{-0.05t} = -e^{-0.05t}$ When $t = 0$, $dT/dt = -1$ When $T = 40$, $40 = 30 + 20e^{-0.05t}$ $\Rightarrow e^{-0.05t} = ½$ $\Rightarrow -0.05t = \ln ½$ $\Rightarrow t = -20 \ln ½ = 13.86..$ (mins)	B1 M1 A1cao M1 M1 A1cao [6]	50 correct derivative –1 (or 1) substituting $T = 40$ taking lns correctly or trial and improvement – one value above and one below or 13.9 or 13 mins 52 secs or better www condone secs

5 $\int_0^1 \frac{x}{2x+1}\,dx$ let $u = 2x + 1$ $\Rightarrow du = 2dx,\ x = \frac{u-1}{2}$ When $x = 0$, $u = 1$, when $x = 1$, $u = 3$ $= \int_1^3 \frac{\frac{1}{2}(u-1)}{u}\frac{1}{2}\,du = \frac{1}{4}\int_1^3 \frac{u-1}{u}\,du$ $= \frac{1}{4}\int_1^3 (1-\frac{1}{u})\,du$ $= \frac{1}{4}\left[u - \ln u\right]_1^3$ $= \frac{1}{4}[3 - \ln 3 - 1 + \ln 1]$ $= \frac{1}{4}(2 - \ln 3)$	M1 A1 B1 M1 A1 E1 [6]	Substituting $\frac{x}{2x+1} = \frac{u-1}{2u}$ o.e. $\frac{1}{4}\int \frac{u-1}{u}\,du$ o.e. [condone no du] converting limits dividing through by u $\frac{1}{4}\left[u - \ln u\right]$ o.e. – ft their ¼ (only) must be some evidence of substitution
6 $y = \frac{x}{2+3\ln x}$ $\Rightarrow \frac{dy}{dx} = \frac{(2+3\ln x).1 - x.\frac{3}{x}}{(2+3\ln x)^2}$ $= \frac{2+3\ln x - 3}{(2+3\ln x)^2}$ $= \frac{3\ln x - 1}{(2+3\ln x)^2}$ When $\frac{dy}{dx} = 0$, $3\ln x - 1 = 0$ $\Rightarrow \ln x = 1/3$ $\Rightarrow x = e^{1/3}$ $\Rightarrow y = \frac{e^{1/3}}{2+1} = \frac{1}{3}e^{1/3}$	M1 B1 A1 M1 A1cao M1 A1cao [7]	Quotient rule consistent with their derivatives or product rule + chain rule on $(2+3x)^{-1}$ $\frac{d}{dx}(\ln x) = \frac{1}{x}$ soi correct expression their numerator = 0 (or equivalent step from product rule formulation) M0 if denominator = 0 is pursued $x = e^{1/3}$ substituting for their x (correctly) Must be exact: – 0.46… is M1A0
7 $y^2 + y = x^3 + 2x$ $x = 2 \Rightarrow y^2 + y = 12$ $\Rightarrow y^2 + y - 12 = 0$ $\Rightarrow (y-3)(y+4) = 0$ $\Rightarrow y = 3$ or -4. $2y\frac{dy}{dx} + \frac{dy}{dx} = 3x^2 + 2$ $\Rightarrow \frac{dy}{dx}(2y+1) = 3x^2 + 2$ $\Rightarrow \frac{dy}{dx} = \frac{3x^2+2}{2y+1}$ At (2, 3), $\frac{dy}{dx} = \frac{12+2}{6+1} = 2$ At (2, –4), $\frac{dy}{dx} = \frac{12+2}{-8+1} = -2$	M1 A1 A1 M1 A1cao M1 A1 cao A1 cao [8]	Substituting $x = 2$ $y = 3$ $y = -4$ Implicit differentiation – LHS must be correct substituting $x = 2$, $y = 3$ into their dy/dx, but must require both x and one of their y to be substituted 2 –2

Section B

8 (i)	At P, $x \sin 3x = 0$ $\Rightarrow \sin 3x = 0$ $\Rightarrow 3x = \pi$ $\Rightarrow x = \pi/3$	M1 A1 A1cao [3]	$x \sin 3x = 0$ $3x = \pi$ or 180 $x = \pi/3$ or 1.05 or better
(ii)	When $x = \pi/6$, $x \sin 3x = \frac{\pi}{6}\sin\frac{\pi}{2} = \frac{\pi}{6}$ $\Rightarrow$ Q($\pi/6$, $\pi/6$) lies on line $y = x$	E1 [1]	$y = \frac{\pi}{6}$ or $x \sin 3x = x \Rightarrow \sin 3x = 1$ etc. Must conclude in radians, and be exact
(iii)	$y = x\sin 3x$ $\Rightarrow \frac{dy}{dx} = x.3\cos 3x + \sin 3x$ At Q, $\frac{dy}{dx} = \frac{\pi}{6}.3\cos\frac{\pi}{2} + \sin\frac{\pi}{2} = 1$ $=$ gradient of $y = x$ So line touches curve at this point	B1 M1 A1cao M1 A1ft E1 [6]	$d/dx(\sin 3x) = 3\cos 3x$ Product rule consistent with their derivs $3x\cos 3x + \sin 3x$ substituting $x = \pi/6$ into their derivative = 1 ft dep 1st M1 = gradient of $y = x$ (www)
(iv)	Area under curve = $\int_0^{\frac{\pi}{6}} x\sin 3x\,dx$ Integrating by parts, $u = x$, $dv/dx = \sin 3x$ $\Rightarrow v = -\frac{1}{3}\cos 3x$ $\int_0^{\frac{\pi}{6}} x\sin 3x\,dx = \left[-\frac{1}{3}x\cos 3x\right]_0^{\frac{\pi}{6}} + \int_0^{\frac{\pi}{6}} \frac{1}{3}\cos 3x\,dx$ $= -\frac{1}{3}.\frac{\pi}{6}.\cos\frac{\pi}{2} + \frac{1}{3}.0.\cos 0 + \left[\frac{1}{9}\sin 3x\right]_0^{\frac{\pi}{6}}$ $= \frac{1}{9}$ Area under line = $\frac{1}{2}\times\frac{\pi}{6}\times\frac{\pi}{6} = \frac{\pi^2}{72}$ So area required = $\frac{\pi^2}{72} - \frac{1}{9}$ $= \frac{\pi^2 - 8}{72}$ *	 M1 A1cao A1ft M1 A1 B1 E1 [7]	 Parts with $u = x$ $dv/dx = \sin 3x \Rightarrow$ $v = -\frac{1}{3}\cos 3x$ [condone no negative] $\ldots + \left[\frac{1}{9}\sin 3x\right]_0^{\frac{\pi}{6}}$ substituting (correct) limits $\frac{1}{9}$ www $\frac{\pi^2}{72}$ www

9 (i) $f(-x) = \ln[1 + (-x)^2]$ $= \ln[1 + x^2] = f(x)$ Symmetrical about Oy	M1 E1 B1 [3]	If verifies that $f(-x) = f(x)$ using a particular point, allow SCB1 For $f(-x) = \ln(1 + x^2) = f(x)$ allow M1E0 For $f(-x) = \ln(1 +- x^2) = f(x)$ allow M1E0 or 'reflects in Oy', etc
(ii) $y = \ln(1 + x^2)$ let $u = 1 + x^2$ $dy/du = 1/u$, $du/dx = 2x$ $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$ $= \frac{1}{u}.2x = \frac{2x}{1+x^2}$ When $x = 2$, $dy/dx = 4/5$.	M1 B1 A1 A1cao [4]	Chain rule $1/u$ soi
(iii) The function is not one to one for this domain	B1 [1]	Or many to one
(iv)  Domain for $g(x) = 0 \le x \le \ln 10$ $y = \ln(1 + x^2)$ $x \leftrightarrow y$ $x = \ln(1 + y^2)$ $\Rightarrow e^x = 1 + y^2$ $\Rightarrow e^x - 1 = y^2$ $\Rightarrow y = \sqrt{(e^x - 1)}$ so $g(x) = \sqrt{(e^x - 1)}$* *or* $g\,f(x) = g[\ln(1 + x^2)]$ $= \sqrt{e^{\ln(1+x^2)} - 1}$ $= (1 + x^2) - 1$ $= x$	M1 A1 B1 M1 M1 E1 M1 M1 E1 [6]	$g(x)$ is $f(x)$ reflected in $y = x$ Reasonable shape and domain, i.e. no –ve x values, inflection shown, does not cross $y = x$ line Condone y instead of x Attempt to invert function Taking exponentials $g(x) = \sqrt{(e^x - 1)}$* www forming $g\,f(x)$ or $f\,g(x)$ $e^{\ln(1+x^2)} = 1 + x^2$ or $\ln(1 + e^x - 1) = x$ www
(v) $g'(x) = \frac{1}{2}(e^x - 1)^{-1/2}.e^x$ $\Rightarrow g'(\ln 5) = \frac{1}{2}(e^{\ln 5} - 1)^{-1/2}.e^{\ln 5}$ $= \frac{1}{2}(5 - 1)^{-1/2}.5$ $= 5/4$ Reciprocal of gradient at P as tangents are reflections in $y = x$.	B1 B1 M1 E1cao B1 [5]	$\frac{1}{2}u^{-1/2}$ soi $\times e^x$ substituting ln 5 into g′ - must be some evidence of substitution Must have idea of reciprocal. Not 'inverse'.

4753 - Methods for Advanced Mathematics

General Comments

There was a pleasing response to the first C3 paper in the new specification. We saw very sound work from many candidates, and most centres appeared to prepare candidates well for the new specification.

The paper discriminated well across the whole ability range. Full marks were scored infrequently, but there were plenty of candidates scoring over 60 out of 72. Equally, even the weakest candidates managed to find some accessible questions and score over 20 marks.

The work was quite well presented, although there was some evidence of poor use of notation in questions1, 5, 8 and 9. A few candidates ran into time trouble – usually caused by inefficient methods or making several attempts at the same question. However, virtually all students attempted all the questions. Candidates should be advised not to spend too long on low tariff section A questions, and to allow ample time to tackle the high tariff section B questions.

Comments on Individual Questions

Section A

1) A fair number of candidates found this an easy three marks. However, there was a noticeable amount of poor notation in solutions, with statements such as $|3x| = -1$. Nearly all the candidates scored the B1 for $x = -1/3$; far fewer achieved $x = -1$. Some candidates treated the question as if it were an inequality. The 'squaring' method was seen relatively infrequently.

2) This three mark question is technically very easy, but was on the whole poorly done, even by quite competent candidates. The concept of inverse trigonometric functions was not well understood.

3) The concept of composite functions seems to be well understood, and this question was well answered. Virtually all candidates found fg(x) = $\ln(x^3)$, although $(\ln(x))^3$ was seen occasionally; The next A1 for $3\ln x$ was achieved by over half of candidates, and the description of the transformation was good – we required 'stretch', 'scale factor 3' and 'parallel to Oy'.

4) The first and last demands were very well answered – the solution of the exponential equation by taking logarithms is a routine in which candidates are usually well drilled. However, the second part on the initial rate was poorly done, and often omitted. Quite a few candidates differentiated correctly but then failed to substitute t = 0.

5) Integration by substitution is found difficult by weaker candidates, who often fail to substitute for the du – the omission of dx often causes this. However, many candidates successfully achieved the first three marks, but then failed to divide through by u (or did so with errors). Some tried integration by parts here, which works after a couple of applications. When substituting limits to achieve a result given on the paper, it is important that sufficient working is shown.

6) The quotient rule was generally well known, with us and vs in the right places. Plenty of candidates achieved 3 marks for a correct expression for the derivative. Thereafter, marks were squandered by faulty algebraic simplification of the numerator(e.g. spurious cancelling), or failing to evaluate the coordinates exactly in terms of e. Setting the denominator to zero and pursuing this value was penalised.

7) Most candidates solved $y^2 + y - 12 = 0$ correctly, sometimes by inspection or formula rather than by factorising. In the implicit differentiation, $2y\,\mathrm{d}y/\mathrm{d}x + 1 = \ldots$ was quit a common error. Good candidates scored full marks here with little difficulty, however.

Section B

8) This question was found to be the most challenging on the paper, with good solutions to part (iv) being quite rare. Students lost marks at various stages through working in degrees rather than radians.

(i) This part was intended as a simple warm-up for three marks. In the event, it was rather poorly done by a lot of candidates. Many candidates could not get further than $x\sin 3x = 0$; $x = 60$ was quite common.

(ii) Candidates had to conclude in radians before achieving this mark.

(iii) The derivative of sin 3x was well known, and many achieved 3 marks for the correct product rule. Candidates had to mention (or imply) that the gradient of $y = x$ was 1 to achieve the E mark. Substituting degrees instead of radians into the derivative was *not* condoned.

(iv) This was the hardest question on the paper. Many recognised the integration by parts, but then got into sign 'muddles' , or integrated sin 3x as 3cos 3x or cos 3x / 3. Some also got the limits wrong, or failed to find and subtract the area under the line.

9 There were quite a few fairly easy marks to be gained in the question, which was often done quite well.

(i) The property of even functions was well understood. Some equate it with 'even powers' of x; some verify using a single value. However, many gave fully correct algebraic proofs, with the brackets in the correct place. Symmetry about Oy was also well done.

(ii) This was a pretty easy 4 marks for candidates who knew how to differentiate lns. $1/(1 + x^2)$ was a common error, however.

(iii) The condition for an inverse was quite well understood, although 'one to many' instead of 'many to one' was seen on occasions.

(iv) A generous method mark was awarded for recognisable attempts to reflect in $y = x$. To achieve the 'A' mark, candidates needed to discount the negative x domain, and have a 'reasonable' shape, including an inflection and correct gradients at the origin.
Finding the inverse function was generally well done. Weaker candidates tried $1/\mathrm{f}(x)$ or expanded $\ln(1 + x^2) = \ln 1 + \ln x^2$.

(v) Differentiating the inverse function proved to be easier than the original, and then substituting to find the gradient of 5/4 was often successfully negotiated. For the final 'B' mark, candidates need to mention or specify the idea of 'reciprocal' – 'inverse' was not enough.