RECOGNISING ACHIEVEMENT

ADVANCED GCE UNIT **4753/01**

MATHEMATICS (MEI)

Methods for Advanced Mathematics (C3)

THURSDAY 18 JANUARY 2007

Afternoon
Time: 1 hour 30 minutes

Additional materials:
Answer booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.

ADVICE TO CANDIDATES

- Read each question carefully and make sure you know what you have to do before starting your answer.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

Section A (36 marks)

1 Fig.1 shows the graphs of $y = |x|$ and $y = |x-2|+1$. The point P is the minimum point of $y = |x-2|+1$, and Q is the point of intersection of the two graphs.

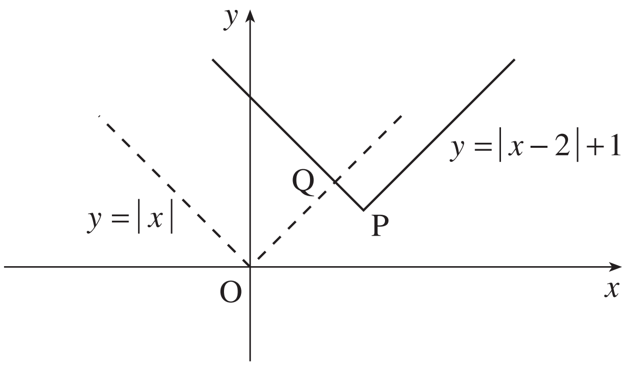

Fig. 1

(i) Write down the coordinates of P. [1]

(ii) Verify that the y-coordinate of Q is $1\frac{1}{2}$. [4]

2 Evaluate $\displaystyle\int_1^2 x^2 \ln x \, dx$, giving your answer in an exact form. [5]

3 The value £V of a car is modelled by the equation $V = Ae^{-kt}$, where t is the age of the car in years and A and k are constants. Its value when new is £10 000, and after 3 years its value is £6000.

(i) Find the values of A and k. [5]

(ii) Find the age of the car when its value is £2000. [2]

4 Use the method of exhaustion to prove the following result.

No 1- or 2-digit perfect square ends in 2, 3, 7 or 8

State a generalisation of this result. [3]

5 The equation of a curve is $y = \dfrac{x^2}{2x+1}$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{2x(x+1)}{(2x+1)^2}$. [4]

(ii) Find the coordinates of the stationary points of the curve. You need not determine their nature. [4]

6 Fig. 6 shows the triangle OAP, where O is the origin and A is the point $(0, 3)$. The point $\mathrm{P}(x, 0)$ moves on the positive x-axis. The point $\mathrm{Q}(0, y)$ moves between O and A in such a way that $\mathrm{AQ} + \mathrm{AP} = 6$.

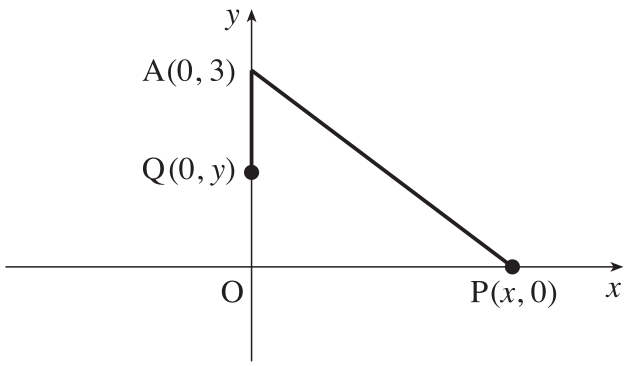

Fig. 6

(i) Write down the length AQ in terms of y. Hence find AP in terms of y, and show that

$$(y+3)^2 = x^2 + 9.$$ [3]

(ii) Use this result to show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y+3}$. [2]

(iii) When $x = 4$ and $y = 2$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 2$. Calculate $\frac{\mathrm{d}y}{\mathrm{d}t}$ at this time. [3]

Section B (36 marks)

7 Fig. 7 shows part of the curve $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = x\sqrt{1+x}$. The curve meets the x-axis at the origin and at the point P.

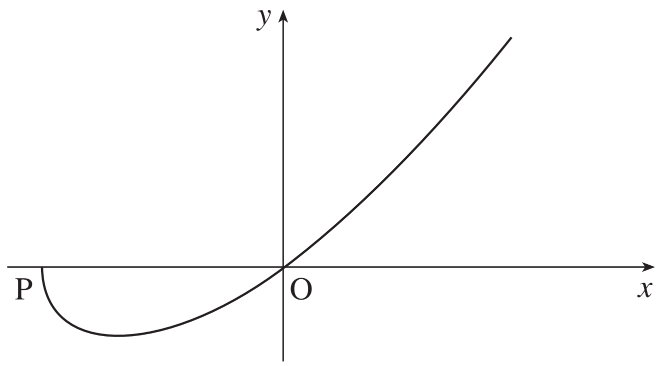

Fig. 7

(i) Verify that the point P has coordinates $(-1, 0)$. Hence state the domain of the function $\mathrm{f}(x)$. [2]

(ii) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2+3x}{2\sqrt{1+x}}$. [4]

(iii) Find the exact coordinates of the turning point of the curve. Hence write down the range of the function. [4]

(iv) Use the substitution $u = 1 + x$ to show that

$$\int_{-1}^{0} x\sqrt{1+x}\,\mathrm{d}x = \int_{0}^{1}\left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)\mathrm{d}u.$$

Hence find the area of the region enclosed by the curve and the x-axis. [8]

8 Fig. 8 shows part of the curve $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = (\mathrm{e}^x - 1)^2 \quad \text{for } x \geqslant 0.$$

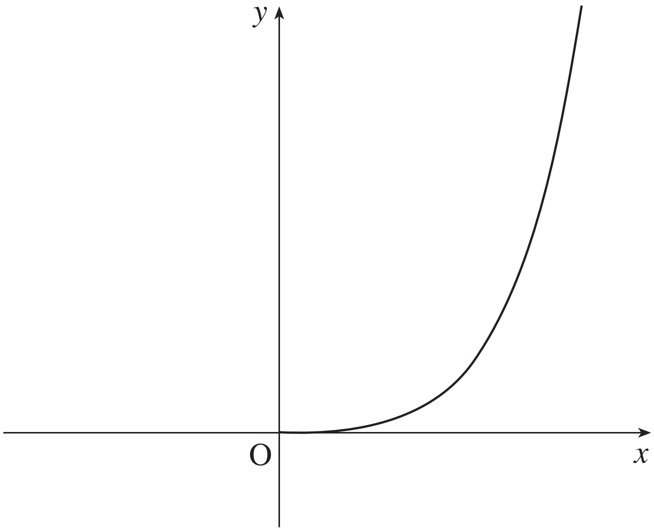

Fig. 8

(i) Find $\mathrm{f}'(x)$, and hence calculate the gradient of the curve $y = \mathrm{f}(x)$ at the origin and at the point $(\ln 2, 1)$. [5]

The function $\mathrm{g}(x)$ is defined by $\mathrm{g}(x) = \ln\left(1 + \sqrt{x}\right)$ for $x \geqslant 0$.

(ii) Show that $\mathrm{f}(x)$ and $\mathrm{g}(x)$ are inverse functions. Hence sketch the graph of $y = \mathrm{g}(x)$.

Write down the gradient of the curve $y = \mathrm{g}(x)$ at the point $(1, \ln 2)$. [5]

(iii) Show that $\displaystyle\int \left(\mathrm{e}^x - 1\right)^2 \mathrm{d}x = \tfrac{1}{2}\mathrm{e}^{2x} - 2\mathrm{e}^x + x + c.$

Hence evaluate $\displaystyle\int_0^{\ln 2} \left(\mathrm{e}^x - 1\right)^2 \mathrm{d}x$, giving your answer in an exact form. [5]

(iv) Using your answer to part **(iii)**, calculate the area of the region enclosed by the curve $y = \mathrm{g}(x)$, the x-axis and the line $x = 1$. [3]

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Mark Scheme 4753
January 2007

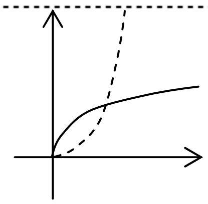
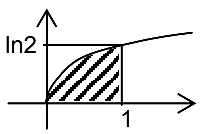
Section A

1 (i) P is (2, 1)	B1	
(ii) $\|x\| = 1\frac{1}{2}$ $\Rightarrow x = (-1\frac{1}{2})$ or $1\frac{1}{2}$ $\|x-2\|+1 = 1\frac{1}{2} \Rightarrow \|x-2\| = \frac{1}{2}$ $\Rightarrow x = (2\frac{1}{2})$ or $1\frac{1}{2}$	M1 A1 M1 E1	allow $x = 1½$ unsupported or $\left\|1\frac{1}{2}-2\right\|+1 = \frac{1}{2}+1 = 1\frac{1}{2}$
or by solving equation directly: $\|x-2\|+1 = \|x\|$ $\Rightarrow$ $2-x+1 = x$ $\Rightarrow$ $x = 1½$ $\Rightarrow$ $y = \|x\| = 1½$	M1 M1 A1 E1 [4]	equating from graph or listing possible cases
2 $\int_1^2 x^2 \ln x dx$ $u = \ln x$ $dv/dx = x^2 \Rightarrow v = \frac{1}{3}x^3$ $= \left[\frac{1}{3}x^3 \ln x\right]_1^2 - \int_1^2 \frac{1}{3}x^3.\frac{1}{x}dx$ $= \frac{8}{3}\ln 2 - \int_1^2 \frac{1}{3}x^2 dx$ $= \frac{8}{3}\ln 2 - \left[\frac{1}{9}x^3\right]_1^2$ $= \frac{8}{3}\ln 2 - \frac{8}{9} + \frac{1}{9}$ $= \frac{8}{3}\ln 2 - \frac{7}{9}$	M1 A1 A1 M1 A1 cao [5]	Parts with $u = \ln x$ $dv/dx = x^2 \Rightarrow v = x^3/3$ $\left[\frac{1}{9}x^3\right]$ substituting limits o.e. – not ln 1
3 (i) When $t = 0$, $V = 10\,000$ $\Rightarrow$ $10\,000 = Ae^0 = A$ When $t = 3$, $V = 6000$ $\Rightarrow$ $6000 = 10\,000\,e^{-3k}$ $\Rightarrow$ $-3k = \ln(0.6) = -0.5108…$ $\Rightarrow$ $k = 0.17(02…)$	M1 A1 M1 M1 A1 [5]	$10\,000 = Ae^0$ $A = 10\,000$ taking lns (correctly) on their exponential equation - not logs unless to base 10 art 0.17 or $-(\ln 0.6)/3$ oe
(ii) $2000 = 10\,000e^{-kt}$ $\Rightarrow$ $-kt = \ln 0.2$ $\Rightarrow$ $t = -\ln 0.2 / k = 9.45$ (years)	M1 A1 [2]	taking lns on correct equation (consistent with their k) allow art 9.5, but not 9.

4 Perfect squares are 0, 1, 4, 9, 16, 25, 36, 49, 64, 81 none of which end in a 2, 3, 7 or 8. Generalisation: no perfect squares end in a 2, 3, 7 or 8.	 M1 E1 B1 [3]	 Listing all 1- and 2- digit squares. Condone absence of 0^2, and listing squares of 2 digit nos (i.e. $0^2 – 19^2$) For extending result to include further square numbers.
5 (i) $y = \frac{x^2}{2x+1}$ $\Rightarrow \frac{dy}{dx} = \frac{(2x+1)2x - x^2.2}{(2x+1)^2}$ $= \frac{2x^2+2x}{(2x+1)^2} = \frac{2x(x+1)}{(2x+1)^2}$ *	 M1 A1 A1 E1 [4]	 Use of quotient rule (or product rule) Correct numerator – condone missing bracket provided it is treated as present Correct denominator www –do not condone missing brackets
(ii) $\frac{dy}{dx} = 0$ when $2x(x + 1) = 0$ $\Rightarrow$ $x = 0$ or -1 $y = 0$ or -1	 B1 B1 B1 B1 [4]	 Must be from correct working: SC -1 if denominator = 0
6(i) QA = $3 - y$, PA = $6 - (3 - y) = 3 + y$ By Pythagoras, $PA^2 = OP^2 + OA^2$ $\Rightarrow (3 + y)^2 = x^2 + 3^2 = x^2 + 9$. *	B1 B1 E1 [3]	 must show some working to indicate Pythagoras (e.g. $x^2 + 3^2$)
(ii) Differentiating implicitly: $2(y+3)\frac{dy}{dx} = 2x$ $\Rightarrow \frac{dy}{dx} = \frac{x}{y+3}$ *	 M1 E1	 Allow errors in RHS derivative (but not LHS) - notation should be correct brackets must be used
or $9 + 6y + y^2 = x^2 + 9$ $\Rightarrow 6y + y^2 = x^2$ $\Rightarrow (6+2y)\frac{dy}{dx} = 2x$ $\Rightarrow \frac{dy}{dx} = \frac{x}{y+3}$	 M1 E1	 Allow errors in RHS derivative (but not LHS) - notation should be correct brackets must be used
or $y = \sqrt{(x^2+ 9)} - 3 \Rightarrow dy/dx = ½(x^2+9)^{-1/2}.2x$ $= \frac{x}{\sqrt{x^2+9}} = \frac{x}{y+3}$	M1 E1	(cao)
(iii) $\frac{dy}{dt} = \frac{dy}{dx}.\frac{dx}{dt}$ $= \frac{4}{2+3} \times 2$ $= \frac{8}{5}$	 M1 A1 A1 [3]	 chain rule (soi)

Section B

7(i) When $x = -1$, $y = -1\sqrt{0} = 0$ Domain $x \geq -1$	E1 B1 [2]	 Not $y \geq -1$
(ii) $\frac{dy}{dx} = x.\frac{1}{2}(1+x)^{-1/2} + (1+x)^{1/2}$ $= ½\,(1 + x)^{-1/2}[x + 2(1 + x)]$ $= \frac{2+3x}{2\sqrt{1+x}}$ *	B1 B1 M1 E1	$x.\frac{1}{2}(1+x)^{-1/2}$ … + $(1 + x)^{1/2}$ taking out common factor or common denominator www
or $u = x + 1 \Rightarrow du/dx = 1$ $\Rightarrow y = (u - 1)u^{1/2} = u^{3/2} - u^{1/2}$ $\Rightarrow \frac{dy}{du} = \frac{3}{2}u^{\frac{1}{2}} - \frac{1}{2}u^{-\frac{1}{2}}$ $\Rightarrow \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \frac{3}{2}(x+1)^{\frac{1}{2}} - \frac{1}{2}(x+1)^{-\frac{1}{2}}$ $= \frac{1}{2}(x+1)^{-\frac{1}{2}}(3x+3-1)$ $= \frac{2+3x}{2\sqrt{1+x}}$ *	M1 A1 M1 E1 [4]	 taking out common factor or common denominator
(iii) $dy/dx = 0$ when $3x + 2 = 0$ $\Rightarrow x = -2/3$ $\Rightarrow y = -\frac{2}{3}\sqrt{\frac{1}{3}}$ Range is $y \geq -\frac{2}{3}\sqrt{\frac{1}{3}}$	M1 A1cao A1 B1 ft [4]	 o.e. not $x \geq -\frac{2}{3}\sqrt{\frac{1}{3}}$ (ft their y value, even if approximate)
(iv) $\int_{-1}^{0} x\sqrt{1+x}\,dx$ let $u = 1 + x$, $du/dx = 1 \Rightarrow du = dx$ when $x = -1$, $u = 0$, when $x = 0$, $u = 1$ $= \int_0^1 (u-1)\sqrt{u}\,du$ $= \int_0^1 (u^{3/2} - u^{1/2})du$ *	 M1 B1 M1 E1	 $du = dx$ or $du/dx = 1$ or $dx/du = 1$ changing limits – allow with no working shown provided limits are present and consistent with dx and du. $(u - 1)\sqrt{u}$ www – condone only final brackets missing, otherwise notation must be correct
$= \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_0^1$ $= \pm\frac{4}{15}$	B1 B1 M1 A1cao [8]	$\frac{2}{5}u^{5/2}, -\frac{2}{3}u^{3/2}$ (oe) substituting correct limits (can imply the zero limit) $\pm\frac{4}{15}$ or ± 0.27 or better, not 0.26

8 (i) $f'(x) = 2(e^x - 1)e^x$ When $x = 0$, f'(0) = 0 When $x = \ln 2$, f'(ln 2) = 2(2 − 1)2 = 4	M1 A1 B1dep M1 A1cao [5]	or $f(x) = e^{2x} - 2e^x + 1$ M1 (or $(e^x)^2 - 2e^x + 1$ plus correct deriv of $(e^x)^2$) $\Rightarrow f'(x) = 2e^{2x} - 2e^x$ A1 derivative must be correct, www $e^{\ln 2} = 2$ soi
(ii) $y = (e^x - 1)^2$ $x \leftrightarrow y$ $x = (e^y - 1)^2$ $\Rightarrow$ $\sqrt{x} = e^y - 1$ $\Rightarrow$ $1 + \sqrt{x} = e^y$ $\Rightarrow$ $y = \ln(1 + \sqrt{x})$	M1 M1 E1	reasonable attempt to invert formula taking lns similar scheme of inverting $y = \ln(1 + \sqrt{x})$
or $gf(x) = g((e^x - 1)^2)$ $= \ln(1 + e^x - 1)$ $= x$	M1 M1 E1	constructing gf or fg $\ln(e^x) = x$ or $e^{\ln(1+\sqrt{x})} = 1 + \sqrt{x}$
 Gradient at (1, ln 2) = ¼	B1 B1ft [5]	reflection in $y = x$ (must have infinite gradient at origin)
(iii) $\int (e^x - 1)^2 dx = \int (e^{2x} - 2e^x + 1)dx$ $= \frac{1}{2}e^{2x} - 2e^x + x + c$ * $\int_0^{\ln 2} (e^x - 1)^2 dx = \left[\frac{1}{2}e^{2x} - 2e^x + x\right]_0^{\ln 2}$ $= \frac{1}{2}e^{2\ln 2} - 2e^{\ln 2} + \ln 2 - (\frac{1}{2} - 2)$ $= 2 - 4 + \ln 2 - \frac{1}{2} + 2$ $= \ln 2 - \frac{1}{2}$	M1 E1 M1 M1 A1 [5]	expanding brackets (condone e^{x^2}) substituting limits $e^{\ln 2} = 2$ used must be exact
(iv)  Area = 1 × ln 2 – (ln 2 – ½) = ½	 M1 B1 A1cao [3]	 subtracting area in (iii) from rectangle rectangle area = 1 × ln 2 must be supported

4753 - Concepts for Advanced Mathematics (C3)

General Comments

The paper proved to be accessible to all but a few candidates, and there were many excellent scripts scoring over 60 marks out of 72, and few scripts scoring less than 20. The standard of presentation was often pleasing, though we did penalise inaccurate notation by withholding E marks for errors in notation, and for lack of brackets – see questions 5(i) and 7(ii) and (iv). Virtually all the candidates appeared to have enough time to complete the paper.

Topics which continue to cause problems to candidates are proof and disproof – the lack of familiarity of question 4, which would not appear to be particularly difficult, seemed to put some candidates off – and the modulus function, which seems to create misunderstandings. The calculus techniques are usually sound, but some aspects of function notation and language, such as inverting, domain and range, are less generally assured.

It should be pointed out that although graph paper is available, and may be requested by candidates, it is almost invariably unnecessary, and can lose time for candidates who try to plot graphs accurately when sketches are requested.

Comments on Individual Questions

Section A

1 Part (i) proved to be an easy 'write down' for most candidates. However, candidates' success with part (ii) was mixed. The easiest approach (which the word 'verify' suggests) is to argue that on $y = |x|$, $y = 1½ \Rightarrow x = 1½$, and then substituting this value into $y = |x-2| + 1$ to verify that $y = 1½$. Students using the direct approach of solving $|x| = |x-2| + 1$ often came unstuck by failing to see that for the point of intersection $y = 2 - x + 1$. Other tactics, such as squaring both sides, are not very satisfactory. Many candidates showed weaknesses in handling moduli.

2 This was found to be one of the harder questions, with a modal score of 3. While there were plenty of completely correct solutions, a substantial minority tried to use $u = x^2$ and $dv/dx = \ln x$, falsely giving $v = 1/x$.
Another source of error was failing to simplify $\int \frac{1}{3}x^3.\frac{1}{x}dx$.

3 This question was very well answered, with the majority of candidates scoring full marks. Virtually all found $A = 10000$, and in general candidates solved the equations by taking lns effectively. There were occasional errors caused by 'fudging' the negative signs.

4 Although only worth 3 marks, this proved to be the least well done question on the paper. A surprising number of candidates ignored the question altogether, perhaps being put off by the 'method of exhaustion' mentioned in the question. Some candidates also interpreted 2-digit perfect squares to mean squares of two-digit numbers, and tested up to 99^2! We wanted to see 0^2 to 9^2 evaluated, together with a comment that none end in 2, 3, 7 or 8. However, we condoned the lack of 0^2, which was quite common, and any other superfluous squares evaluated. Many candidates, for the generalisation, commented on the pattern rather than generalising the given statement to all integers or whole numbers.

5 This proved to be high scoring and straightforward. The given result helped candidates to keep on track. We withheld the 'E' mark if brackets were missing in the numerator of the quotient rule. Weaker candidates missed the $x = 0$ solution in part (ii), or set numerator equal to denominator. We deducted a mark (when one was earned) for equating the denominator to zero.

6 Many candidates made a meal of part (i), often finding AQ = $3 - y$ but failing to conclude that AP = $3 + y$. Part (ii) was quite well done – many successfully expanded the brackets first, and some deduced the result from $y = \sqrt{(x^2 + 9)} - 3$. Part (iii) was also generally well done, provided it was initiated by a correct chain rule.

Section B

7 The given results made many of the marks in this question accessible, and very few candidates failed to score half marks for this question – the modal mark was 14.

(i) Verifying the coordinates of P was a straightforward mark, but the domain $x \geq -1$ was less well known.

(ii) The product rule was well done, but the algebra required to show the given result proved to be beyond weaker candidates.

(iii) The turning point coordinates were correctly derived by the majority of candidates; the range was sometimes omitted, and y was occasionally approximated.

(iv) We required to see $du/dx = 1$, or $du = dx$, limits consistent with dx and du, and brackets in $(u - 1)\sqrt{u}$, to secure the E mark. The integration was done well – negative answers were accepted.

8 Parts of this question were more challenging for candidates than the rest of the paper. In particular, the last part required geometrical insight which was often lacking, and maximum marks were rare. The modal mark was 11 out of 18.

(i) Most candidates differentiated correctly, either by expanding the brackets or using the chain rule. Many went on to evaluate the gradients at $x = 0$ and $x = \ln 2$ correctly. Errors in the derivative leading to a fortuitous zero gradient at the origin were not allowed follow through.

(ii) Although there was some confusion between $f^{-1}(x)$ and $f'(x)$, $f(-x)$ or $1/f(x)$, most candidates attempted to invert $y = (e^x - 1)^2$ and many completed this work successfully. Other approaches were to show that $fg(x)$ or $gf(x)$ equals x, or use of a flow diagram, which was accepted provided the order of operations was clear. We did not expect particularly accurate graphs, provided the $g(x)$ appeared to be a reasonable reflection in $y = x$; however, we wanted to see the curve touching the y-axis, and graphs which dipped substantially below the horizontal were penalised. The gradient at $(1, \ln 2)$ was often calculated by differentiating. A fairly common error was $-\frac{1}{4}$.

(iii) Even with the help of the result given, many candidates failed to expand brackets before integrating term by term, and tried substitution or parts, usually without success. This sometimes wasted a lot of time. The final result often suffered from losing negatives after substituting the limits.

(iv) This proved to be a good discriminator for A grade candidates. Some tried to evaluate the integral directly, which is difficult though not impossible. Others guessed that the integral equalled the result in (iii) or its reciprocal. Only the best candidates were able to sort out the geometry successfully. Unsupported answers, presumably obtained by graphics calculator, gained no credit.