OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS (MEI) 4753/01

Methods for Advanced Mathematics (C3)

Candidates answer on the Answer Booklet

OCR Supplied Materials:
- 8 page Answer Booklet
- MEI Examination Formulae and Tables (MF2)

Other Materials Required:
None

Wednesday 20 January 2010
Afternoon

Duration: 1 hour 30 minutes

4753010

INSTRUCTIONS TO CANDIDATES

- Write your name clearly in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully and make sure that you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

Turn over

Section A (36 marks)

1 Solve the equation $e^{2x} - 5e^x = 0$. **[4]**

2 The temperature T in degrees Celsius of water in a glass t minutes after boiling is modelled by the equation $T = 20 + be^{-kt}$, where b and k are constants. Initially the temperature is 100 °C, and after 5 minutes the temperature is 60 °C.

(i) Find b and k. **[4]**

(ii) Find at what time the temperature reaches 50 °C. **[2]**

3 **(i)** Given that $y = \sqrt[3]{1 + 3x^2}$, use the chain rule to find $\dfrac{dy}{dx}$ in terms of x. **[3]**

(ii) Given that $y^3 = 1 + 3x^2$, use implicit differentiation to find $\dfrac{dy}{dx}$ in terms of x and y. Show that this result is equivalent to the result in part **(i)**. **[4]**

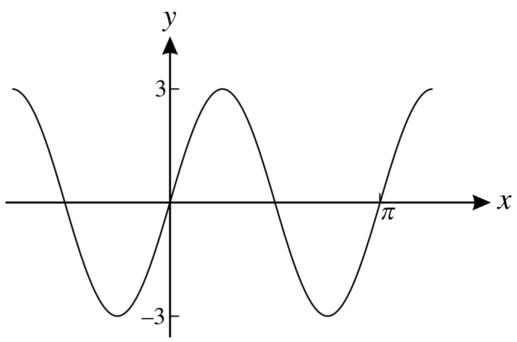
4 Evaluate the following integrals, giving your answers in exact form.

(i) $\displaystyle\int_0^1 \frac{2x}{x^2 + 1}\, dx.$ **[3]**

(ii) $\displaystyle\int_0^1 \frac{2x}{x + 1}\, dx.$ **[5]**

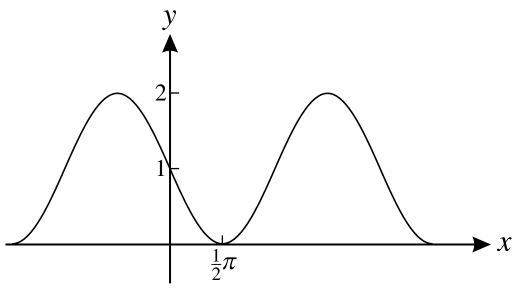
5 The curves in parts **(i)** and **(ii)** have equations of the form $y = a + b\sin cx$, where a, b and c are constants. For each curve, find the values of a, b and c.

(i)



[2]

(ii)



[2]

6 Write down the conditions for $f(x)$ to be an odd function and for $g(x)$ to be an even function.

Hence prove that, if $f(x)$ is odd and $g(x)$ is even, then the composite function $gf(x)$ is even. **[4]**

7 Given that $\arcsin x = \arccos y$, prove that $x^2 + y^2 = 1$. [Hint: let $\arcsin x = \theta$.] **[3]**

Section B (36 marks)

8 Fig. 8 shows part of the curve $y = x \cos 3x$.

The curve crosses the x-axis at O, P and Q.

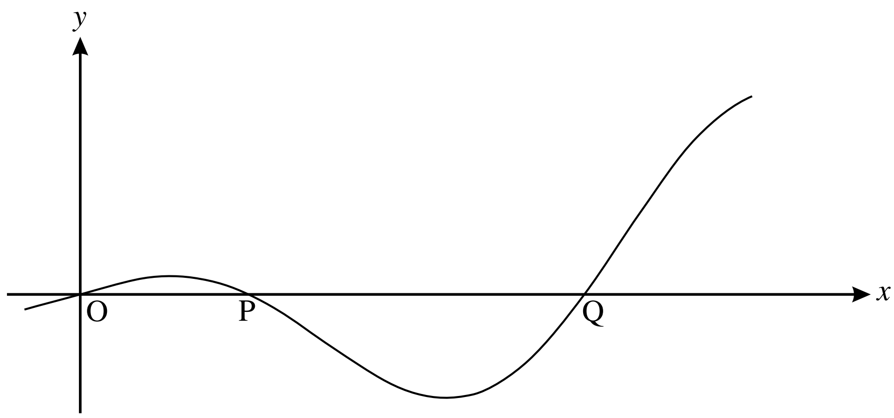


Fig. 8

(i) Find the exact coordinates of P and Q. **[4]**

(ii) Find the exact gradient of the curve at the point P.

Show also that the turning points of the curve occur when $x \tan 3x = \frac{1}{3}$. **[7]**

(iii) Find the area of the region enclosed by the curve and the x-axis between O and P, giving your answer in exact form. **[6]**

[Question 9 is printed overleaf.]

9 Fig. 9 shows the curve $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \dfrac{2x^2 - 1}{x^2 + 1}$ for the domain $0 \leqslant x \leqslant 2$.

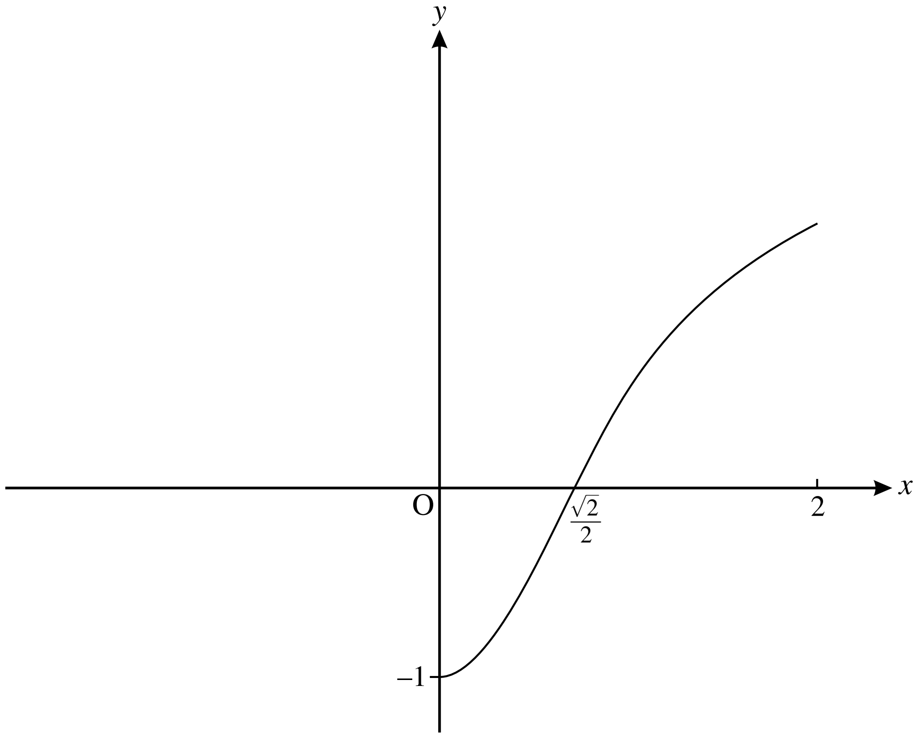


Fig. 9

(i) Show that $\mathrm{f}'(x) = \dfrac{6x}{(x^2 + 1)^2}$, and hence that $\mathrm{f}(x)$ is an increasing function for $x > 0$. **[5]**

(ii) Find the range of $\mathrm{f}(x)$. **[2]**

(iii) Given that $\mathrm{f}''(x) = \dfrac{6 - 18x^2}{(x^2 + 1)^3}$, find the maximum value of $\mathrm{f}'(x)$. **[4]**

The function $\mathrm{g}(x)$ is the inverse function of $\mathrm{f}(x)$.

(iv) Write down the domain and range of $\mathrm{g}(x)$. Add a sketch of the curve $y = \mathrm{g}(x)$ to a copy of Fig. 9. **[4]**

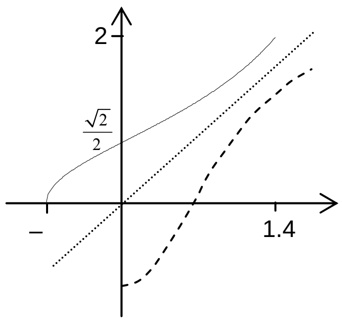
(v) Show that $\mathrm{g}(x) = \sqrt{\dfrac{x+1}{2-x}}$. **[4]**

4753 (C3) Methods for Advanced Mathematics

1 $e^{2x} - 5e^x = 0$ $\Rightarrow$ $e^x(e^x - 5) = 0$ $\Rightarrow$ $e^x = 5$ $\Rightarrow$ $x = \ln 5$ or 1.6094	M1 M1 A1 A1 [4]	factoring out e^x or dividing $e^{2x} = 5e^x$ by e^x $e^{2x} / e^x = e^x$ ln 5 or 1.61 or better, mark final answer –1 for additional solutions, e.g. $x = 0$
or $\ln(e^{2x}) = \ln(5e^x)$ $\Rightarrow$ $2x = \ln 5 + x$ $\Rightarrow$ $x = \ln 5$ or 1.6094	M1 A1 A1 A1 [4]	taking lns on $e^{2x} = 5e^x$ $2x$, $\ln 5 + x$ ln 5 or 1.61 or better, mark final answer –1 for additional solutions, e.g. $x = 0$
2 (i) When $t = 0$, $T = 100$ $\Rightarrow$ $100 = 20 + b$ $\Rightarrow$ $b = 80$ When $t = 5$, $T = 60$ $\Rightarrow$ $60 = 20 + 80e^{-5k}$ $\Rightarrow$ $e^{-5k} = ½$ $\Rightarrow$ $k = \ln 2 / 5 = 0.139$	M1 A1 M1 A1 [4]	substituting $t = 0$, $T = 100$ cao substituting $t = 5$, $T = 60$ 1/5 ln 2 or 0.14 or better
(ii) $50 = 20 + 80e^{-kt}$ $\Rightarrow$ $e^{-kt} = 3/8$ $\Rightarrow$ $t = \ln(8/3) / k = 7.075$ mins	M1 A1 [2]	Re-arranging and taking lns correctly – ft their b and k answers in range 7 to 7.1
3(i) $\frac{dy}{dx} = \frac{1}{3}(1+3x^2)^{-2/3}.6x$ $= 2x(1+3x^2)^{-2/3}$	M1 B1 A1 [3]	chain rule $1/3\, u^{-2/3}$ or $\frac{1}{3}(1+3x^2)^{-2/3}$ o.e but must be '2' (not 6/3) mark final answer
(ii) $3y^2 \frac{dy}{dx} = 6x$ $\Rightarrow$ $dy/dx = 6x/3y^2$ $= \frac{2x}{(1+3x^2)^{2/3}} = 2x(1+3x^2)^{-2/3}$	M1 A1 A1 E1 [4]	$3y^2 \frac{dy}{dx}$ $= 6x$ if deriving $2x(1+3x^2)^{-2/3}$, needs a step of working

4(i) $\int_0^1 \frac{2x}{x^2+1}\,dx = \left[\ln(x^2+1)\right]_0^1$ $= \ln 2$	M2 A1 [3]	$[\ln(x^2+1)]$ cao (must be exact)
or let $u = x^2 + 1$, $du = 2x\,dx$ $\Rightarrow \int_0^1 \frac{2x}{x^2+1}\,dx = \int_1^2 \frac{1}{u}\,du$ $= \left[\ln u\right]_1^2$ $= \ln 2$	M1 A1 A1 [3]	$\int \frac{1}{u}\,du$ or $\left[\ln(1+x^2)\right]_0^1$ with correct limits cao (must be exact)
(ii) $\int_0^1 \frac{2x}{x+1}\,dx = \int_0^1 \frac{2x+2-2}{x+1}\,dx = \int_0^1 (2-\frac{2}{x+1})\,dx$ $= \left[2x - 2\ln(x+1)\right]_0^1$ $= 2 - 2\ln 2$	M1 A1, A1 A1 A1 [5]	dividing by $(x + 1)$ 2, $-2/(x+1)$
or $\int_0^1 \frac{2x}{x+1}\,dx$ let $u = x + 1$, $\Rightarrow du = dx$ $= \int_1^2 \frac{2(u-1)}{u}\,du$ $= \int_1^2 (2-\frac{2}{u})\,du$ $= \left[2u - 2\ln u\right]_1^2$ $= 4 - 2\ln 2 - (2 - 2\ln 1)$ $= 2 - 2\ln 2$	M1 B1 M1 A1 A1 [5]	substituting $u = x + 1$ and $du = dx$ (or $du/dx = 1$) *and* correct limits used for u or x $2(u-1)/u$ dividing through by u $2u - 2\ln u$ allow ft on $(u-1)/u$ (i.e. with 2 omitted) o.e. cao (must be exact)
5 (i) $a = 0$, $b = 3$, $c = 2$	B(2,1,0)	or $a = 0$, $b = -3$, $c = -2$
(ii) $a = 1$, $b = -1$, $c = 1$ *or* $a = 1$, $b = 1$, $c = -1$	B(2,1,0) [4]	
6 $f(-x) = -f(x)$, $g(-x) = g(x)$ $g\,f(-x) = g\,[-f(x)]$ $= g\,f(x)$ $\Rightarrow$ g f is even	B1B1 M1 E1 [4]	condone f and g interchanged forming $gf(-x)$ or $gf(x)$ and using $f(-x) = -f(x)$ www
7 Let $\arcsin x = \theta$ $\Rightarrow x = \sin\theta$ $\theta = \arccos y \Rightarrow y = \cos\theta$ $\sin^2\theta + \cos^2\theta = 1$ $\Rightarrow x^2 + y^2 = 1$	 M1 M1 E1 [3]	

8(i) At P, $x \cos 3x = 0$ $\Rightarrow$ $\cos 3x = 0$ $\Rightarrow$ $3x = \pi/2, 3\pi/2$ $\Rightarrow$ $x = \pi/6, \pi/2$ So P is $(\pi/6, 0)$ and Q is $(\pi/2, 0)$	M1 M1 A1 A1 [4]	or verification $3x = \pi/2, (3\pi/2\ldots)$ dep both Ms condone degrees here
(ii) $\frac{dy}{dx} = -3x\sin 3x + \cos 3x$ At P, $\frac{dy}{dx} = -\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2} = -\frac{\pi}{2}$ At TPs $\frac{dy}{dx} = -3x\sin 3x + \cos 3x = 0$ $\Rightarrow$ $\cos 3x = 3x \sin 3x$ $\Rightarrow$ $1 = 3x \sin 3x / \cos 3x = 3x \tan 3x$ $\Rightarrow$ $x \tan 3x = 1/3$ *	M1 B1 A1 M1 A1cao M1 E1 [7]	Product rule $d/dx (\cos 3x) = -3 \sin 3x$ cao (so for $dy/dx = -3x\sin3x$ allow B1) mark final answer substituting their $-\pi/6$ (must be rads) $-\pi/2$ $dy/dx = 0$ and $\sin 3x / \cos 3x = \tan 3x$ used www
(iii) $A = \int_0^{\pi/6} x\cos 3x\,dx$ Parts with $u = x$, $dv/dx = \cos 3x$ $du/dx = 1$, $v = 1/3 \sin 3x$ $\Rightarrow$ $A = \left[\frac{1}{3}x\sin 3x\right]_0^{\frac{\pi}{6}} - \int_0^{\pi/6}\frac{1}{3}\sin 3x\,dx$ $= \left[\frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x\right]_0^{\frac{\pi}{6}}$ $= \frac{\pi}{18} - \frac{1}{9}$	B1 M1 A1 A1 M1dep A1 cao [6]	Correct integral and limits (soi) – ft their P, but must be in radians can be without limits dep previous A1. substituting correct limits, dep 1st M1: ft their P provided in radians o.e. but must be exact

9(i) $f'(x)=\frac{(x^2+1)4x-(2x^2-1)2x}{(x^2+1)^2}$ $=\frac{4x^3+4x-4x^3+2x}{(x^2+1)^2}=\frac{6x}{(x^2+1)^2}$ * When $x > 0$, $6x > 0$ and $(x^2+1)^2 > 0$ $\Rightarrow$ $f'(x) > 0$	M1 A1 E1 M1 E1 [5]	Quotient or product rule correct expression www attempt to show or solve f'(x) > 0 numerator > 0 and denominator > 0 or, if solving, $6x > 0 \Rightarrow x > 0$
(ii) f(2) = $\frac{8-1}{4+1}=1\frac{2}{5}$ Range is $-1 \le y \le 1\frac{2}{5}$	B1 B1 [2]	 must be ≤, y or f(x)
(iii) f'(x) max when f''(x) = 0 $\Rightarrow$ $6 - 18x^2 = 0$ $\Rightarrow$ $x^2 = 1/3$, $x = 1/\sqrt{3}$ $\Rightarrow$ $f'(x)=\frac{6/\sqrt{3}}{(1\frac{1}{3})^2}=\frac{6}{\sqrt{3}}.\frac{9}{16}=\frac{9\sqrt{3}}{8}=1.95$	 M1 A1 M1 A1 [4]	 (±)1/√3 oe (0.577 or better) substituting 1/√3 into f'(x) 9√3/8 o.e. or 1.95 or better (1.948557..)
(iv) Domain is $-1 < x < 1\frac{2}{5}$ Range is $0 \le y \le 2$  	B1 B1 M1 A1 cao [4]	ft their 1.4 but not $x \ge -1$ or $0 \le g(x) \le 2$ (not f) Reasonable reflection in $y = x$ from (–1, 0) to (1.4, 2), through (0, √2/2) allow omission of one of –1, 1.4, 2, √2/2
(v) $y=\frac{2x^2-1}{x^2+1}$ $x \leftrightarrow y$ $x=\frac{2y^2-1}{y^2+1}$ $\Rightarrow$ $xy^2 + x = 2y^2 - 1$ $\Rightarrow$ $x + 1 = 2y^2 - xy^2 = y^2(2 - x)$ $\Rightarrow$ $y^2=\frac{x+1}{2-x}$ $\Rightarrow$ $y=\sqrt{\frac{x+1}{2-x}}$ *	 M1 M1 M1 E1 [4]	(could start from g) Attempt to invert clearing fractions collecting terms in y^2 and factorising www

4753 Methods for Advanced Mathematics (Written Paper)

General Comments

This paper proved to be accessible to nearly all candidates. However, there were some questions, such as 3, 6 and 7(ii) which tested the abler candidates. Fewer than usual candidates scored full marks – the final part of question 7 proved the main stumbling block – but, equally, virtually all candidates scored above 20. There was no evidence of lack of time to complete the paper.

With reference to recent examiner's reports, it was pleasing to note that fewer candidates were using graph paper for their sketch in question 3. In general, some candidates seem to be insufficiently aware of the significance of words such as 'verify' (see question 8(ii)), and 'hence': most candidates missed the significance of this in questions 6 and 7(ii). Candidates also need to be clear what is meant by exact answers, or they will lose marks in this paper.

In general, the calculus topics continue to be well answered, albeit with some sloppy notation used in integration, with modulus, proof and inverse trigonometric functions being less securely understood. The standard of presentation varied from chaotic to exemplary.

Comments on Individual Questions

Section A

1 There were many good responses to this opening question, but there were a variety of incorrect approaches. One was to attempt to take logs before rearranging to $\mathrm{e}^{2x} = 5\mathrm{e}^{x}$. This led them to the error of $\ln(\mathrm{e}^{2x} - 5\mathrm{e}^{x}) = \ln \mathrm{e}^{2x} - \ln 5\mathrm{e}^{x}$. Other mistakes were in factoring out e^{x}, or inability to simplify $\ln 5\mathrm{e}^{x}$ correctly.

2 This was extremely well done, with most candidates scoring full marks. A few candidates lost a mark for leaving *k* as –(1/5)ln(1/2). Over-accurate answers were not penalised in this instance. The laws of logarithms were generally correctly applied here.

3 This proved to be a straightforward test of the chain rule and implicit differentiation, with many candidates scoring full marks. Occasional index errors were made, and some candidates failed to simplify their answer to part (i), leaving it as $6x(1+3x^2)^{-2/3}/3$. The implicit differentiation in part (ii) was generally well done, and most candidates who had correctly done part (i) correctly showed the equivalence of their answers.

4 Part (i) was usually done correctly, with many candidates spotting the $f'(x)/f(x)$ form of the integrand and writing $\ln(x^2 + 1)$ directly rather than using integration by substitution. However, part (ii) proved considerably more searching. A large number of candidates tried integration by parts without success. Those who used the correct substitution either made substitution errors or failed to split the fraction before integrating. We also required some evidence for d*x* = d*u* for the first M1 – this was usually correctly given.

5 Part (i) was more successfully answered than part (ii), with $c = 1/3$ the most common error. In part (ii), many tried to use a translation to account for the reflection of the sine curve, which did not fit in with the required form of the function.

6 Most candidates gave correct algebraic definitions of odd and even function, often accompanied by redundant geometric descriptions, though some candidates failed to put the brackets and negative signs in the right places. However, the second request was rarely convincing – most candidates just substituted particular functions to verify the result, and were unable to deal with the abstraction of the general argument.

7 This proof question was also generally not well done, despite the hint. Getting from $\theta = \arccos x$ to $x = \cos\theta$ appeared to be considerably harder than might be expected. Some candidates correctly deduced that $x = \cos\theta$ and $y = \sin\theta$ but failed to convince by appearing to argue that $x^2 + y^2 = 1 \Rightarrow \cos^2\theta + \sin^2\theta = 1$. Indeed, many candidates seem to fail to understand the concept of direction of argument in proof. Common errors seen were $\arcsin x = 1/\sin x$ and $\cos x = \sin y$.

Section B

8 This question tested differentiation and integration techniques as applied to a trigonometric function. Some candidates worked in degrees which was condoned in part (i) but inappropriate thereafter. In general, many candidates scored well.

(i) This was perhaps the hardest part of the question for some candidates. Solutions in degrees were accepted, but quite a few made errors in dealing with the '3' in $\cos 3x = 0$, and in finding the second intercept Q. Although condoned in this particular case, students should avoid making statements like $P = \pi/6$, and some wrote the coordinates the wrong way round.

(ii) The product rule was done well. In finding the gradient at P, some worked in degrees and gained M0. The given result on turning points was reasonably well done, though some wrote $\cos 3x / \sin 3x = \tan 3x$.

(iii) Integration by parts was well known, with fewer candidates seeming to make errors with $dv/dx = \cos 3x \Rightarrow v = 1/3 \sin 3x$. Limits from part (i), if expressed in radians, were followed through up to the final 'A' mark: errors in evaluating $\cos 0 = 1$ were not uncommon.

9 This question relied on a good understanding of function language (domain, range, inverse etc.) and required careful use of notation to obtain full marks. The calculus in part (i) was well done, but that in part (iii) concerning the second derivative was less secure.

(i) The quotient rule was done well, though omitting necessary brackets forfeited the final 'E' mark. The deduction that f was increasing required justification that both the top and bottom of the derivative fraction were positive – many just verified it was positive for individual points.

(ii) The range required the student to evaluate f(2) to find the maximum: some assumed the domain was unbounded. We required correct inequalities (≤ not <) and y or $f(x)$ for the range – some gave this as 2.4, which was disallowed.

(iii) This was less successfully done – setting the second derivative to zero to find the maximum of the derivative was less familiar.

(iv) We insisted on correct inequalities and consistent use of x, y or $g(x)$ in the statement of domain and range, but followed through their answer to the range of f, provided this was bounded. For the sketch, most of these were recognisable as attempts to reflect in $y = x$, but needed three out of four of -1, $\sqrt{2}/2$, 1.4 and 2 to gain the A1. Some candidates misinterpreted 'on a copy of Fig. 9' as meaning that this copy should have been provided – we marked question papers which were included with scripts. On-screen marking will facilitate the provision of copies of figures such as this.