OCR RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS (MEI) 4753/01

Methods for Advanced Mathematics (C3)

QUESTION PAPER

Candidates answer on the printed answer book.

OCR supplied materials:
- Printed answer book 4753/01
- MEI Examination Formulae and Tables (MF2)

Other materials required:
- Scientific or graphical calculator

Wednesday 19 January 2011
Afternoon

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

These instructions are the same on the printed answer book and the question paper.

- The question paper will be found in the centre of the printed answer book.
- Write your name, centre number and candidate number in the spaces provided on the printed answer book. Please write clearly and in capital letters.
- **Write your answer to each question in the space provided in the printed answer book.** Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

This information is the same on the printed answer book and the question paper.

- The number of marks is given in brackets **[ ]** at the end of each question or part question on the question paper.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- The printed answer book consists of **12** pages. The question paper consists of **8** pages. Any blank pages are indicated.

INSTRUCTION TO EXAMS OFFICER / INVIGILATOR

- Do not send this question paper for marking; it should be retained in the centre or destroyed.

Section A (36 marks)

1 Given that $y = \sqrt[3]{1 + x^2}$, find $\frac{dy}{dx}$. **[4]**

2 Solve the inequality $|2x + 1| \geqslant 4$. **[4]**

3 The area of a circular stain is growing at a rate of $1\,\text{mm}^2$ per second. Find the rate of increase of its radius at an instant when its radius is 2 mm. **[5]**

4 Use the triangle in Fig. 4 to prove that $\sin^2\theta + \cos^2\theta = 1$. For what values of θ is this proof valid? **[3]**

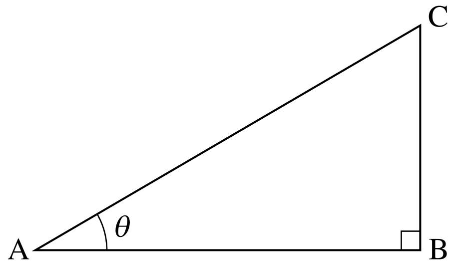


Fig. 4

5 **(i)** On a single set of axes, sketch the curves $y = e^x - 1$ and $y = 2e^{-x}$. **[3]**

(ii) Find the exact coordinates of the point of intersection of these curves. **[5]**

6 A curve is defined by the equation $(x + y)^2 = 4x$. The point $(1,\ 1)$ lies on this curve.

By differentiating implicitly, show that $\frac{dy}{dx} = \frac{2}{x + y} - 1$.

Hence verify that the curve has a stationary point at $(1,\ 1)$. **[4]**

7 Fig. 7 shows the curve $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = 1 + 2\arctan x$, $x \in \mathbb{R}$. The scales on the x- and y-axes are the same.

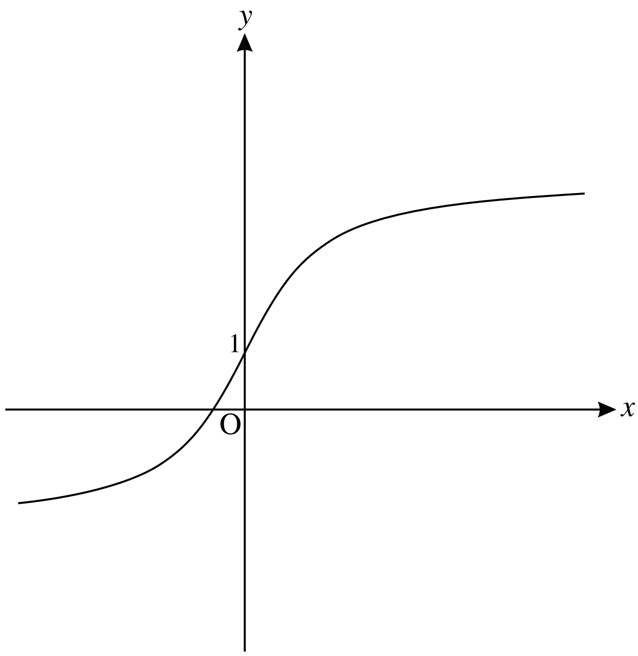


Fig. 7

(i) Find the range of f, giving your answer in terms of π. **[3]**

(ii) Find $\mathrm{f}^{-1}(x)$, and add a sketch of the curve $y = \mathrm{f}^{-1}(x)$ to the copy of Fig. 7. **[5]**

Section B (36 Marks)

8 **(i)** Use the substitution $u = 1 + x$ to show that

$$\int_0^1 \frac{x^3}{1+x}\,dx = \int_a^b \left(u^2 - 3u + 3 - \frac{1}{u}\right) du,$$

where a and b are to be found.

Hence evaluate $\displaystyle\int_0^1 \frac{x^3}{1+x}\,dx$, giving your answer in exact form. **[7]**

Fig. 8 shows the curve $y = x^2 \ln(1 + x)$.

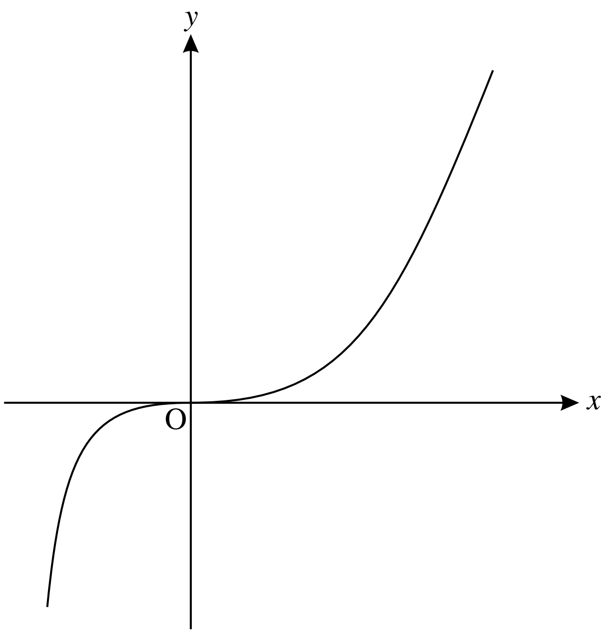


Fig. 8

(ii) Find $\frac{dy}{dx}$.

Verify that the origin is a stationary point of the curve. **[5]**

(iii) Using integration by parts, and the result of part **(i)**, find the exact area enclosed by the curve $y = x^2 \ln(1 + x)$, the x-axis and the line $x = 1$. **[6]**

9 Fig. 9 shows the curve $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \dfrac{1}{\cos^2 x}$, $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$, together with its asymptotes $x = \frac{1}{2}\pi$ and $x = -\frac{1}{2}\pi$.

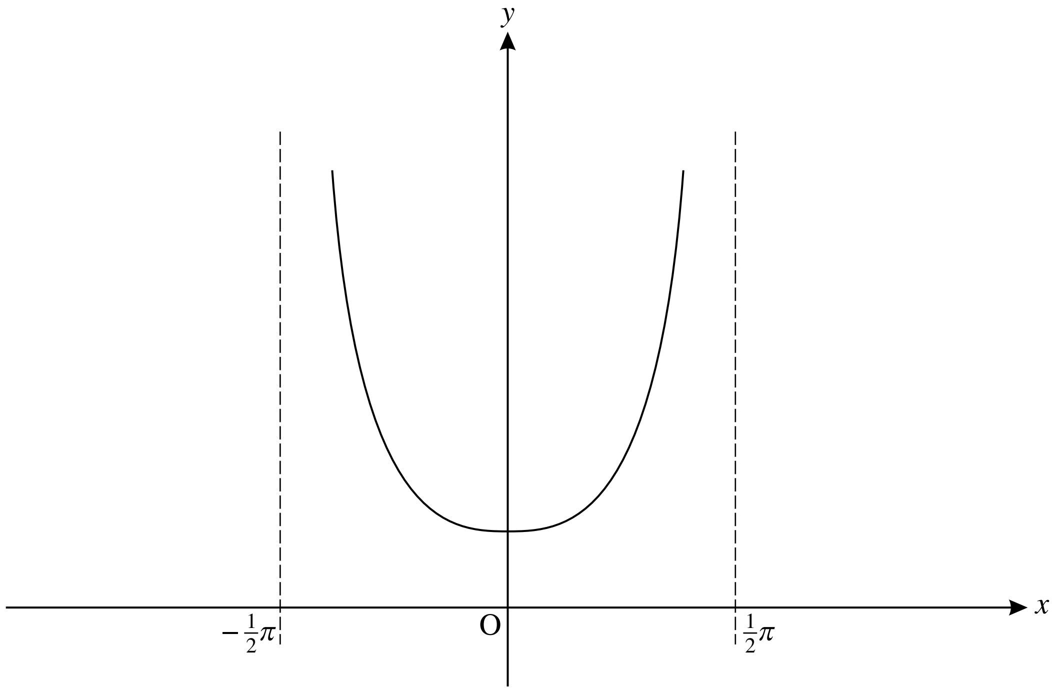


Fig. 9

(i) Use the quotient rule to show that the derivative of $\dfrac{\sin x}{\cos x}$ is $\dfrac{1}{\cos^2 x}$. **[3]**

(ii) Find the area bounded by the curve $y = \mathrm{f}(x)$, the x-axis, the y-axis and the line $x = \frac{1}{4}\pi$. **[3]**

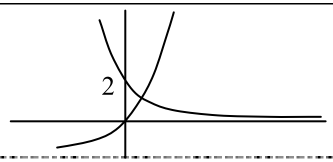
The function $\mathrm{g}(x)$ is defined by $\mathrm{g}(x) = \frac{1}{2}\mathrm{f}\left(x + \frac{1}{4}\pi\right)$.

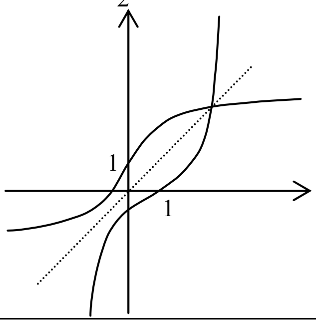
(iii) Verify that the curves $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$ cross at $(0, 1)$. **[3]**

(iv) State a sequence of two transformations such that the curve $y = \mathrm{f}(x)$ is mapped to the curve $y = \mathrm{g}(x)$.

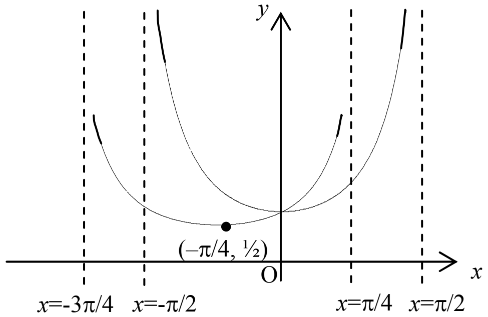
On the copy of Fig. 9, sketch the curve $y = \mathrm{g}(x)$, indicating clearly the coordinates of the minimum point and the equations of the asymptotes to the curve. **[8]**

(v) Use your result from part **(ii)** to write down the area bounded by the curve $y = \mathrm{g}(x)$, the x-axis, the y-axis and the line $x = -\frac{1}{4}\pi$. **[1]**

1 $y=\sqrt[3]{1+x^2}=(1+x^2)^{1/3}$ $\Rightarrow \dfrac{dy}{dx}=\dfrac{1}{3}(1+x^2)^{-\frac{2}{3}}.2x$ $=\dfrac{2}{3}x(1+x^2)^{-\frac{2}{3}}$	M1 M1 B1 A1 [4]	$(1+x^2)^{1/3}$ chain rule $(1/3)\,u^{-2/3}$ (soi) cao, mark final answer	Do not allow MR for square root their $dy/du \times du/dx$ (available for wrong indices) no ft on ½ index oe e.g. $\dfrac{2x(1+x^2)^{-\frac{2}{3}}}{3}$, $\dfrac{2x}{3\sqrt[3]{(1+x^2)^2}}$, etc but must combine 2 with 1/3.
2 $\lvert 2x+1\rvert \geq 4$ $\Rightarrow$ $2x+1\geq 4 \Rightarrow x \geq 1½$ or $2x+1 \leq -4 \Rightarrow x \leq -2½$	M1 A1 M1 A1 [4]	allow M1 for 1½ seen allow M1 for −2½ seen	Same scheme for other methods, e.g. squaring, graphing Penalise both > and < once only. −1 if both correct but final ans expressed incorrectly, e.g $-2½ \geq x \geq 1½$ or $1½ \leq x \leq -2½$ (or even $-2½ \leq x \leq 1½$ from previously correct work) e.g. SC3
3 $A=\pi r^2$ $\Rightarrow$ $dA/dr = 2\pi r$ When $r=2$, $dA/dr = 4\pi$, $dA/dt = 1$ $\dfrac{dA}{dt}=\dfrac{dA}{dr}.\dfrac{dr}{dt}$ $\Rightarrow$ $1 = 4\pi.dr/dt$ $\Rightarrow$ $dr/dt = 1/4\pi = 0.0796$ (mm/s)	M1A1 A1 M1 A1 [5]	$2\pi r$ soi (at any stage) chain rule (o.e) cao: 0.08 or better condone truncation	M1A0 if incorrect notation, e.g. dy/dx, dr/dA, if seen. $2r$ is M1A0 must be dA/dr (soi) and dA/dt any correct form stated with relevant variables , e.g. $\dfrac{dr}{dt}=\dfrac{dr}{dA}.\dfrac{dA}{dt}$, $\dfrac{dr}{dt}=\dfrac{dr}{dA}/\dfrac{dt}{dA}$, etc. allow $1/4\pi$ but mark final answer
4 $\sin\theta = BC/AC$, $\cos\theta = AB/AC$ $AB^2 + BC^2 = AC^2$ $\Rightarrow$ $(AB/AC)^2 + (BC/AC)^2 = 1$ $\Rightarrow$ $\cos^2\theta + \sin^2\theta = 1$ Valid for $(0° <)\ \theta < 90°$	M1 A1 B1 [3]	or a/b, c/b condone taking AC = 1 Must use Pythagoras allow ≤, or 'between 0 and 90' or < 90 allow < π/2 or 'acute'	allow o/h, a/h etc if clearly marked on triangle. but must be stated arguing backwards unless ⇔ used A0
5(i) 	B1 B1 B1 [3]	shape of $y = e^x - 1$ and through O shape of $y = 2e^{-x}$ through (0, 2) (not (2,0))	for first and second B1s graphs must include negative x values condone no asymptote $y = -1$ shown asymptotic to x-axis (shouldn't cross)
(ii) $e^x - 1 = 2e^{-x}$ $\Rightarrow$ $e^{2x} - e^x = 2$ $\Rightarrow$ $(e^x)^2 - e^x - 2 = 0$ $\Rightarrow$ $(e^x - 2)(e^x + 1) = 0$ $\Rightarrow$ $e^x = 2$ (or −1) $\Rightarrow$ $x = \ln 2$ $\Rightarrow$ $y = 1$	M1 M1 B1 B1 B1cao [5]	equating re-arranging into a quadratic in $e^x = 0$ stated www www www	 allow one error but must have $e^{2x} = (e^x)^2$ (soi) award even if not from quadratic method (i.e. by 'fitting') provided www allow for unsupported answers, provided www need not have used a quadratic, provided www

6 $(x+y)^2 = 4x$ $\Rightarrow$ $2(x+y)(1+\frac{dy}{dx}) = 4$ $\Rightarrow$ $1+\frac{dy}{dx} = \frac{4}{2(x+y)} = \frac{2}{x+y}$ $\Rightarrow$ $\frac{dy}{dx} = \frac{2}{x+y} - 1$ *	M1 A1 A1	Implicit differentiation of LHS correct expression = 4 www (**AG**)	Award no marks for solving for y and attempting to differentiate allow one error but must include dy/dx ignore superfluous dy/dx = … for M1, and for both A1s if not pursued condone missing brackets A0 if missing brackets in earlier working
or $x^2 + 2xy + y^2 = 4x$ $\Rightarrow$ $2x + 2x\frac{dy}{dx} + 2y + 2y\frac{dy}{dx} = 4$ $\Rightarrow$ $\frac{dy}{dx}(2x+2y) = 4 - 2x - 2y$ $\Rightarrow$ $\frac{dy}{dx} = \frac{4}{2x+2y} - 1 = \frac{2}{x+y} - 1$ *	M1dep A1 A1	Implicit differentiation of LHS dep correct expansion correct expression = 4 (oe after re-arrangement) www (**AG**)	allow 1 error provided 2xdy/dx and 2ydy/dx are correct, but must expand $(x+y)^2$ correctly for M1 (so $x^2 + y^2 = 4x$ is M0) ignore superfluous dy/dx = … for M1, and for both A1s if not pursued A0 if missing brackets in earlier working
When $x = 1$, $y = 1$, $\frac{dy}{dx} = \frac{2}{1+1} - 1 = 0$ *	B1 [4]	(**AG**) oe (e.g. from $x + y = 2$)	or e.g $2/(x+y) - 1 = 0 \Rightarrow x + y = 2$, $\Rightarrow 4 = 4x$, $\Rightarrow x = 1, y = 1$ (oe)
7 (i) bounds $-\pi + 1$, $\pi + 1$ $\Rightarrow$ $-\pi + 1 < f(x) < \pi + 1$	B1B1 B1cao [3]	or … < y < … or $(-\pi + 1, \pi + 1)$	not … < x < …, not 'between …'
(ii) $y = 2\arctan x + 1$ $x \leftrightarrow y$ $x = 2\arctan y + 1$ $\Rightarrow$ $\frac{x-1}{2} = \arctan y$ $\Rightarrow$ $y = \tan(\frac{x-1}{2}) \Rightarrow f^{-1}(x) = \tan(\frac{x-1}{2})$ 	M1 A1 A1 B1 B1 [5]	attempt to invert formula or $\frac{y-1}{2} = \arctan x$ reasonable reflection in $y = x$ (1, 0) intercept indicated.	one step is enough, i.e. $y - 1 = 2\arctan x$ or $x - 1 = 2\arctan y$ need not have interchanged x and y at this stage allow $y = …$ curves must cross on $y = x$ line if present (or close enough to imply intention) curves shouldn't touch or cross in the third quadrant

8(i) $\int_0^1 \frac{x^3}{1+x}\,dx$ let $u = 1+x,\ du = dx$ when $x = 0$, $u = 1$, when $x = 1$, $u = 2$ $= \int_1^2 \frac{(u-1)^3}{u}\,du$ $= \int_1^2 \frac{(u^3-3u^2+3u-1)}{u}\,du$ $= \int_1^2 (u^2-3u+3-\frac{1}{u})\,du$ * $\int_0^1 \frac{x^3}{1+x}\,dx = \left[\frac{1}{3}u^3-\frac{3}{2}u^2+3u-\ln u\right]_1^2$ $= (\frac{8}{3}-6+6-\ln 2)-(\frac{1}{3}-\frac{3}{2}+3-\ln 1)$ $= \frac{5}{6}-\ln 2$	B1 B1 M1 A1dep B1 M1 A1cao [7]	$a = 1, b = 2$ $(u-1)^3/u$ expanding (correctly) dep du = dx (o.e.) **AG** $\left[\frac{1}{3}u^3-\frac{3}{2}u^2+3u-\ln u\right]$ substituting correct limits dep integrated must be exact – must be 5/6	seen anywhere, e.g. in new limits e.g. du/dx = 1, condone missing dx's and du's, allow du = 1 upper – lower; may be implied from 0.140… must have evaluated ln 1 = 0
(ii) $y = x^2 \ln(1+x)$ ⇒ $\frac{dy}{dx} = x^2.\frac{1}{1+x}+2x.\ln(1+x)$ $= \frac{x^2}{1+x}+2x\ln(1+x)$ When $x = 0$, dy/dx = 0 + 0.ln 1 = 0 (⇒ Origin is a stationary point)	M1 B1 A1 M1 A1cao [5]	Product rule d/dx (ln(1 + x)) = 1/(1 + x) cao (oe) mark final ans substituting x = 0 into **correct** deriv www	 or d/dx (ln u) = 1/u where u = 1 + x ln1+x is A0 when x = 0, dy/dx = 0 with no evidence of substituting M1A0 but condone missing bracket in ln(1+x)
(iii) $A = \int_0^1 x^2 \ln(1+x)\,dx$ let $u = \ln(1+x)$, $dv/dx = x^2$ $\frac{du}{dx} = \frac{1}{1+x}$, $v = \frac{1}{3}x^3$ ⇒ $A = \left[\frac{1}{3}x^3 \ln(1+x)\right]_0^1 - \int_0^1 \frac{1}{3}\frac{x^3}{1+x}\,dx$ $= \frac{1}{3}\ln 2-(\frac{5}{18}-\frac{1}{3}\ln 2)$ $= \frac{1}{3}\ln 2-\frac{5}{18}+\frac{1}{3}\ln 2$ $= \frac{2}{3}\ln 2-\frac{5}{18}$	B1 M1 A1 B1 B1ft A1 [6]	Correct integral and limits parts correct $= \frac{1}{3}\ln 2 - ...$ … – 1/3 (result from part (i)) cao	condone no dx, limits (and integral) can be implied by subsequent work u, du/dx, dv/dx and v all correct (oe) condone missing brackets condone missing bracket, can re-work from scratch oe e.g. $= \frac{12\ln 2-5}{18}, \frac{1}{3}\ln 4-\frac{5}{18}$, etc but must have evaluated ln 1 =0 Must combine the two ln terms

9(i) $\frac{d}{dx}(\frac{\sin x}{\cos x}) = \frac{\cos x.\cos x - \sin x.(-\sin x)}{\cos^2 x}$ $= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$ *	M1 A1 A1 [3]	Quotient (or product) rule (**AG**)	product rule: $\frac{1}{\cos x}.\cos x + \sin x(-\frac{1}{\cos^2 x})(-\sin x)$ but must show evidence of using chain rule on 1/cos x (or d/dx (sec x) = sec x tan x used)
(ii) Area = $\int_0^{\pi/4} \frac{1}{\cos^2 x} dx$ $= [\tan x]_0^{\pi/4}$ = tan(π/4) – tan 0 = 1	B1 M1 A1 [3]	correct integral and limits (soi) $[\tan x]$ or $\left[\frac{\sin x}{\cos x}\right]$	condone no dx; limits can be implied from subsequent work unsupported scores M0
(iii) f(0) = 1/cos²(0) = 1 g(x) = 1/2cos²(x + π/4) g(0) = 1/2cos²(π/4) = 1 (⇒ f and g meet at (0, 1))	B1 M1 A1 [3]	must show evidence	or f(π/4) = 1/cos²(π/4) = 2 so g(0) = ½ f(π/4) = 1
(iv) Translation in x-direction through –π/4 Stretch in y-direction scale factor ½ 	M1 A1 M1 A1 B1ft B1ft B1 B1dep [8]	must be in x-direction, or $\begin{pmatrix}-\pi/4\\0\end{pmatrix}$ must be in y-direction asymptotes correct min point (–π/4, ½) curves intersect on y-axis correct curve, dep B3, with asymptote lines indicated and correct, and TP in correct position	'shift' or 'move' for 'translation' M1 A0; $\begin{pmatrix}-\pi/4\\0\end{pmatrix}$ alone SC1 'contract' or 'compress' or 'squeeze' for 'stretch' M1A0; 'enlarge' M0 stated or on graph; condone no x = …, ft π/4 to right only (viz. –π/4, 3π/4) stated or on graph; ft π/4 to right only (viz. (π/4, ½)) 'y-values halved', or 'x-values reduced by π/4, are M0 (not geometric transformations), but for M1 condone mention of x- and y- values provided transformation words are used.
(v) Same as area in (ii), but stretched by s.f. ½ . So area = ½ .	B1ft [1]	½ area in (ii)	or $\int_{-\pi/4}^{0} g(x)\,dx = \frac{1}{2}\int_{-\pi/4}^{0} \frac{1}{\cos^2(x+\pi/4)} dx = \frac{1}{2}[\tan(x+\pi/4]_{-\pi/4}^{0}$ = ½ allow unsupported