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1)

$$y = \sqrt[3]{1+x^2}$$
$$y = (1+x^2)^{\frac{1}{3}}$$
$$\frac{dy}{dx} = \frac{1}{3}(1+x^2)^{-\frac{2}{3}} \times 2x$$
$$= \frac{2}{3}x(1+x^2)^{-\frac{2}{3}}$$

2)

$$|2x+1| \geqslant 4$$

Either

$$2x+1 \geqslant 4$$
$$2x \geqslant 4-1$$
$$2x \geqslant 3$$
$$x \geqslant \frac{3}{2}$$

or

$$2x+1 \leq -4$$
$$2x \leq -4-1$$
$$2x \leq -5$$
$$x \leq -\frac{5}{2}$$

Solution:

Either $x \geqslant \frac{3}{2}$

or $x \leq -\frac{5}{2}$

3 Circle $A = \pi r^2$

$$\frac{dA}{dr} = 2\pi r$$

Given that $\frac{dA}{dt} = 1 \text{ mm}^2\text{s}^{-1}$

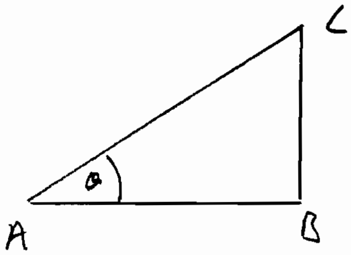
Find $\frac{dr}{dt}$ when $r = 2\text{mm}$

$$\frac{dr}{dt} = \frac{dr}{dA} \times \frac{dA}{dt}$$
$$= \frac{1}{\frac{dA}{dr}} \times \frac{dA}{dt}$$
$$= \frac{1}{2\pi r} \times 1 \quad \text{mms}^{-1}$$

When $r = 2\text{mm}$

$$\frac{dr}{dt} = \frac{1}{2\times\pi\times 2} \times 1 \quad \text{mms}^{-1}$$
$$\frac{dr}{dt} = 0.079577 \quad \text{mms}^{-1}$$
$$= 0.0796 \text{ mms}^{-1} \text{ to 3 s.f.}$$

4)



$$\sin\theta = \frac{BC}{AC}, \quad \cos\theta = \frac{AB}{AC}$$
$$\tan\theta = \frac{BC}{AB}$$
$$\sin^2\theta + \cos^2\theta = \frac{BC^2}{AC^2} + \frac{AB^2}{AC^2}$$

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4 cont)

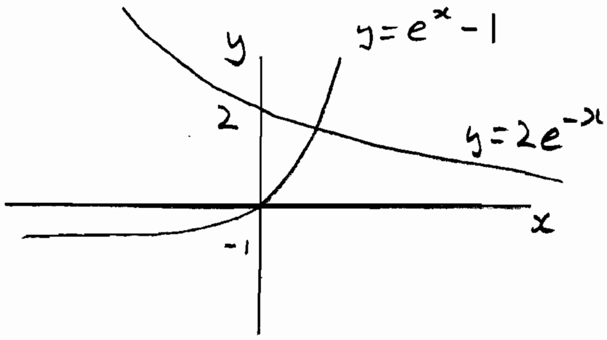
$$= \frac{BC^2 + AB^2}{AC^2}$$

But by Pythagoras $BC^2 + AB^2 = AC^2$

$\therefore \sin^2\theta + \cos^2\theta = \frac{AC^2}{AC^2} = 1$

Proof valid for $0^\circ < \theta < 90^\circ$

5) i)



ii)

$y = e^x - 1$ ①

$y = 2e^{-x}$ ②

Subst for y in ①

$$2e^{-x} = e^x - 1$$

Multiply by e^x

$$2 = e^{2x} - e^x$$

$$e^{2x} - e^x - 2 = 0$$

$$(e^x + 1)(e^x - 2) = 0$$

Either $(e^x + 1) = 0$

$e^x = -1$ ✗ impossible

or $e^x - 2 = 0$

$$e^x = 2$$

$\Rightarrow x = \ln 2$

When $x = \ln 2$

$$y = e^{\ln 2} - 1$$

$$y = 2 - 1 = 1$$

Point of intersection is

$$(\ln 2, 1)$$

6) $(x+y)^2 = 4x$

$$2(x+y)\left(1 + \frac{dy}{dx}\right) = 4$$

$$(x+y)\left(1 + \frac{dy}{dx}\right) = 2$$

$$(x+y) + (x+y)\frac{dy}{dx} = 2$$

$$(x+y)\frac{dy}{dx} = 2 - (x+y)$$

$$\frac{dy}{dx} = \frac{2 - (x+y)}{(x+y)}$$

$$\frac{dy}{dx} = \frac{2}{x+y} - 1$$

Given $(1,1)$ on curve

At $(1,1)$, $\frac{dy}{dx} = \frac{2}{1+1} - 1$

$$= \frac{2}{2} - 1 = 0$$

$\therefore$ a st. pt. at $(1,1)$

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Section B

7)

i) $y = f(x) = 1 + 2\tan^{-1}x$

$x \in \mathbb{R}$

$\tan^{-1}\infty = \frac{\pi}{2}$, $\tan^{-1}-\infty = -\frac{\pi}{2}$

Range of $f(x)$

$1 - 2\times\frac{\pi}{2} < f(x) < 1 + 2\times\frac{\pi}{2}$

$1 - \pi < f(x) < 1 + \pi$

ii) $y = 1 + 2\tan^{-1}x$

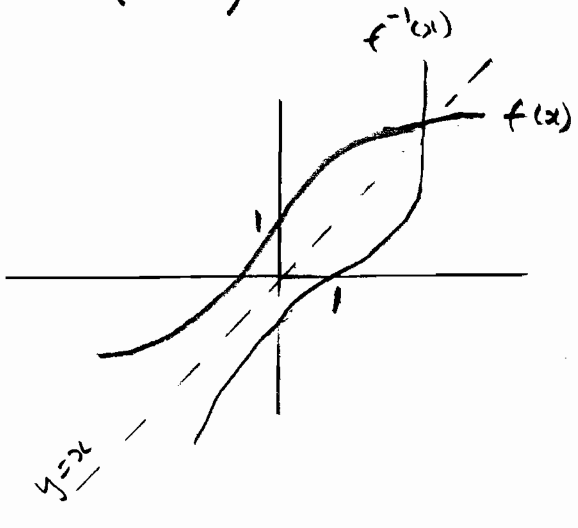
swap variables and rearrange

$x = 1 + 2\tan^{-1}y$

$x - 1 = 2\tan^{-1}y$

$\frac{x-1}{2} = \tan^{-1}y$

$\tan\left(\frac{x-1}{2}\right) = y$



$f^{-1}(x)$ is $f(x)$ reflected in line $y = x$

8)

i) $\int_0^1 \frac{x^3}{1+x}\,dx$

Let $u = 1 + x \Rightarrow x = u - 1$

$\frac{du}{dx} = 1 \Rightarrow du = dx$

When $x = 1$, $u = 1 + 1 = 2$

when $x = 0$, $u = 1 + 0 = 1$

$\int_0^1 \frac{x^3}{1+x}\,dx = \int_1^2 \frac{(u-1)^3}{u}\,du$

$= \int_1^2 \frac{u^3 - 3u^2 + 3u - 1}{u}\,du$

(using binomial expansion 1, 3, 3, 1)

$= \int_1^2 \left(u^2 - 3u + 3 - \frac{1}{u}\right) du$

$= \left[\frac{u^3}{3} - \frac{3u^2}{2} + 3u - \ln u\right]_1^2$

$= \left(\frac{8}{3} - 6 + 6 - \ln 2\right) - \left(\frac{1}{3} - \frac{3}{2} + 3 - 0\right)$

$= \frac{8}{3} - \ln 2 - \frac{1}{3} + \frac{3}{2} - 3$

$= \frac{5}{6} - \ln 2$

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8 ii)

$$y = x^2 \ln(1+x)$$

$$\frac{dy}{dx} = x^2 \times \frac{1}{1+x} + \ln(1+x) \times 2x$$

$$= \frac{x^2}{1+x} + 2x\ln(1+x)$$

When $x = 0$

$$\frac{dy}{dx} = \frac{0}{1} + 2 \times 0 \times \ln 1$$

$$= 0$$

$\therefore$ st pt when $x = 0$

when $x = 0$, $y = 0$

$\therefore$ st. pt. at origin

8 iii)

$$\text{Area} = \int_0^1 x^2 \ln(1+x)\,dx$$

Let $u = \ln(1+x)$, Let $\frac{dv}{dx} = x^2$

$\Rightarrow \frac{du}{dx} = \frac{1}{1+x}$ $\Rightarrow v = \frac{x^3}{3}$

Using $\int u \frac{dv}{dx} = uv - \int v \frac{du}{dx}$

$$\int_0^1 x^2 \ln(1+x)\,dx$$

$$= \left[\frac{x^3}{3}\ln(1+x)\right]_0^1 - \int_0^1 \frac{x^3}{3(1+x)}\,dx$$

$$= \left[\frac{x^3}{3}\ln(1+x)\right]_0^1 - \frac{1}{3}\int_0^1 \frac{x^3}{1+x}\,dx$$

$$= \left[\frac{1}{3}\ln 2 - 0\right] - \frac{1}{3}\left(\frac{5}{6} - \ln 2\right)$$

$$= \frac{1}{3}\ln 2 - \frac{5}{18} + \frac{1}{3}\ln 2$$

$$= \frac{2}{3}\ln 2 - \frac{5}{18}$$

9) $y = f(x) = \frac{1}{\cos^2 x}$

$$-\frac{\pi}{2} < x < \frac{\pi}{2}$$

i) $\frac{d}{dx}\frac{u}{v} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos x \times \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

ii) $\text{Area} = \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x}\,dx$

$$= \left[\frac{\sin x}{\cos x}\right]_0^{\frac{\pi}{4}}$$

$$= \left[\tan x\right]_0^{\frac{\pi}{4}} = \tan\frac{\pi}{4} - \tan 0$$

$$= 1$$

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9 iii) $g(x) = \frac{1}{2} f\left(x + \frac{\pi}{4}\right)$

For $y = f(x)$,

when $x = 0$, $y = \frac{1}{\cos^2 0} = 1$

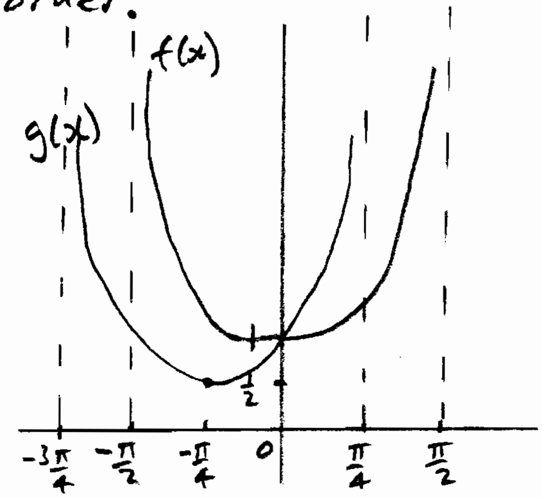
For $y = g(x)$

when $x = 0$, $y = \frac{1}{2} \times \frac{1}{\cos^2 \frac{\pi}{4}}$

$y = \frac{1}{2} \times \frac{1}{\frac{1}{2}} = 1$

$\therefore (0, 1)$ is on both $f(x)$ and $g(x)$

9 iv) $f(x)$ is mapped to $g(x)$ by a translation by $\begin{pmatrix} -\frac{\pi}{4} \\ 0 \end{pmatrix}$ and a one-way stretch parallel to y-axis scale factor $\frac{1}{2}$. These could be carried out in either order.



Min point of $g(x)$ is $\left(-\frac{\pi}{4}, \frac{1}{2}\right)$

Asymptotes of $g(x)$ are

$x = -\frac{3\pi}{4}$, $x = \frac{\pi}{4}$

9 v) The required area corresponds to the area found under $f(x)$ in part (ii). The translation does not affect the calculation but the stretch vertically by scale factor $\frac{1}{2}$ means that the size of the area is scaled by a multiple of $\frac{1}{2}$.

Area $= 1 \times \frac{1}{2} = \frac{1}{2}$