OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS (MEI) 4753/01

Methods for Advanced Mathematics (C3)

QUESTION PAPER

Candidates answer on the printed answer book.

OCR supplied materials:
- Printed answer book 4753/01
- MEI Examination Formulae and Tables (MF2)

Other materials required:
- Scientific or graphical calculator

Monday 20 June 2011
Morning

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

These instructions are the same on the printed answer book and the question paper.

- The question paper will be found in the centre of the printed answer book.
- Write your name, centre number and candidate number in the spaces provided on the printed answer book. Please write clearly and in capital letters.
- **Write your answer to each question in the space provided in the printed answer book.** Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

This information is the same on the printed answer book and the question paper.

- The number of marks is given in brackets **[ ]** at the end of each question or part question on the question paper.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- The printed answer book consists of **16** pages. The question paper consists of **4** pages. Any blank pages are indicated.

INSTRUCTION TO EXAMS OFFICER / INVIGILATOR

- Do not send this question paper for marking; it should be retained in the centre or destroyed.

Section A (36 marks)

1 Solve the equation $|2x - 1| = |x|$. **[4]**

2 Given that $f(x) = 2\ln x$ and $g(x) = e^x$, find the composite function $gf(x)$, expressing your answer as simply as possible. **[3]**

3 **(i)** Differentiate $\dfrac{\ln x}{x^2}$, simplifying your answer. **[4]**

(ii) Using integration by parts, show that $\displaystyle\int \frac{\ln x}{x^2}\,dx = -\frac{1}{x}(1 + \ln x) + c$. **[4]**

4 The height h metres of a tree after t years is modelled by the equation

$$h = a - be^{-kt},$$

where a, b and k are positive constants.

(i) Given that the long-term height of the tree is 10.5 metres, and the initial height is 0.5 metres, find the values of a and b. **[3]**

(ii) Given also that the tree grows to a height of 6 metres in 8 years, find the value of k, giving your answer correct to 2 decimal places. **[3]**

5 Given that $y = x^2\sqrt{1+4x}$, show that $\dfrac{dy}{dx} = \dfrac{2x(5x+1)}{\sqrt{1+4x}}$. **[5]**

6 A curve is defined by the equation $\sin 2x + \cos y = \sqrt{3}$.

(i) Verify that the point P$\left(\frac{1}{6}\pi, \frac{1}{6}\pi\right)$ lies on the curve. **[1]**

(ii) Find $\dfrac{dy}{dx}$ in terms of x and y.

Hence find the gradient of the curve at the point P. **[5]**

7 **(i)** Multiply out $(3^n + 1)(3^n - 1)$. **[1]**

(ii) Hence prove that if n is a positive integer then $3^{2n} - 1$ is divisible by 8. **[3]**

Section B (36 marks)

8

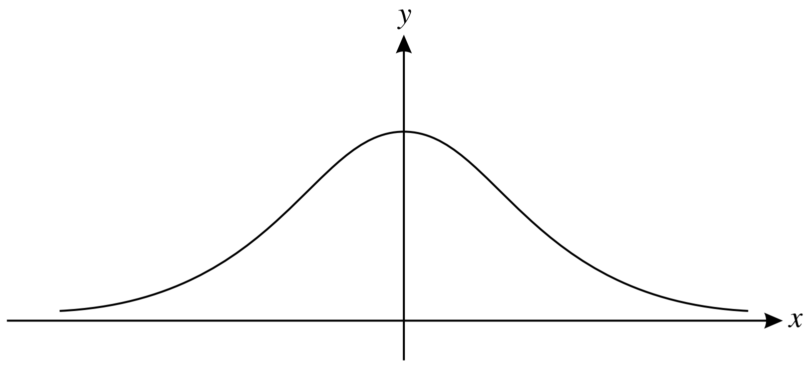

Fig. 8

Fig. 8 shows the curve $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \dfrac{1}{e^x + e^{-x} + 2}$.

(i) Show algebraically that $\mathrm{f}(x)$ is an even function, and state how this property relates to the curve $y = \mathrm{f}(x)$. **[3]**

(ii) Find $\mathrm{f}'(x)$. **[3]**

(iii) Show that $\mathrm{f}(x) = \dfrac{e^x}{(e^x + 1)^2}$. **[2]**

(iv) Hence, using the substitution $u = e^x + 1$, or otherwise, find the exact area enclosed by the curve $y = \mathrm{f}(x)$, the x-axis, and the lines $x = 0$ and $x = 1$. **[5]**

(v) Show that there is only one point of intersection of the curves $y = \mathrm{f}(x)$ and $y = \frac{1}{4}e^x$, and find its coordinates. **[5]**

[Question 9 is printed overleaf.]

9 Fig. 9 shows the curve $y = \mathrm{f}(x)$. The endpoints of the curve are P $(-\pi, 1)$ and Q $(\pi, 3)$, and $\mathrm{f}(x) = a + \sin bx$, where a and b are constants.

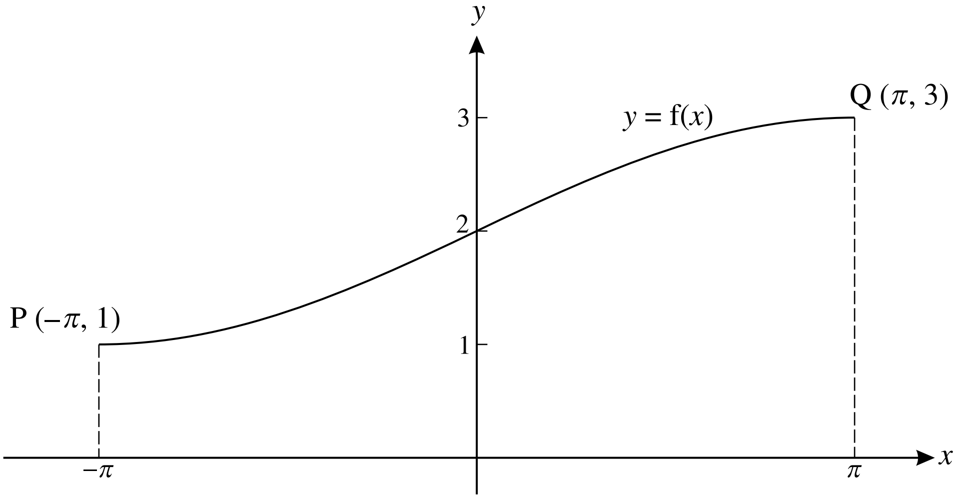

Fig. 9

(i) Using Fig. 9, show that $a = 2$ and $b = \frac{1}{2}$. **[3]**

(ii) Find the gradient of the curve $y = \mathrm{f}(x)$ at the point $(0, 2)$.

Show that there is no point on the curve at which the gradient is greater than this. **[5]**

(iii) Find $\mathrm{f}^{-1}(x)$, and state its domain and range.

Write down the gradient of $y = \mathrm{f}^{-1}(x)$ at the point $(2, 0)$. **[6]**

(iv) Find the area enclosed by the curve $y = \mathrm{f}(x)$, the x-axis, the y-axis and the line $x = \pi$. **[4]**

1 $\lvert 2x-1\rvert = \lvert x\rvert$ $\Rightarrow$ $2x-1=x$, $x=1$ or $-(2x-1)=x$, $x=1/3$	M1A1 M1A1 [4]	www www, or $2x-1=-x$ must be exact for A1 (e.g. not 0.33, but allow $0.\dot{3}$) condone doing both equalities in one line e.g. $-x=2x-1=x$, etc	allow unsupported answers or from graph or squaring $\Rightarrow 3x^2-4x+1=0$ M1 $\Rightarrow (3x-1)(x-1)=0$ M1 factorising, formula or comp. square $\Rightarrow x=1$, 1/3 A1 A1 allow M1 for sign errors in factorisation –1 if more than two solutions offered, but isw inequalities
2 $\text{gf}(x)=e^{2\ln x}$ $=e^{\ln x^2}$ $=x^2$	M1 M1 A1 [3]	Forming gf(x) (soi)	Doing fg: $2\ln(e^x)=2x$ SC1 Allow x^2 (but not $2x$) unsupported
3(i) $\dfrac{dy}{dx}=\dfrac{x^2.\frac{1}{x}-\ln x.2x}{x^4}$ $=\dfrac{x-2x\ln x}{x^4}$ $=\dfrac{1-2\ln x}{x^3}$	M1 B1 A1 A1 [4]	quotient rule with $u=\ln x$ and $v=x^2$ d/dx (ln x) = 1/x soi correct expression (o.e.) o.e. cao, mark final answer, but must have divided top and bottom by x	Consistent with their derivatives. $u\text{d}v \pm v\text{d}u$ in the quotient rule is M0 Condone $\ln x.2x=\ln 2x^2$ for this A1 (provided $\ln x.2x$ is shown) e.g. $\dfrac{1}{x^3}-\dfrac{2\ln x}{x^3}$, $x^{-3}-2x^{-3}\ln x$
or $\dfrac{dy}{dx}=-2x^{-3}\ln x+x^{-2}(\dfrac{1}{x})$ $=-2x^{-3}\ln x+x^{-3}$	M1 B1 A1 A1 [4]	product rule with $u=x^{-2}$ and $v=\ln x$ d/dx (ln x) = 1/x soi correct expression o.e. cao, mark final answer, must simplify the $x^{-2}.(1/x)$ term.	or vice-versa
(ii) $\int \dfrac{\ln x}{x^2}dx$ let $u=\ln x$, du/dx = 1/x dv/dx = $1/x^2$, $v=-x^{-1}$ $=-\dfrac{1}{x}\ln x+\int \dfrac{1}{x}.\dfrac{1}{x}dx$ $=-\dfrac{1}{x}\ln x+\int \dfrac{1}{x^2}dx$ $=-\dfrac{1}{x}\ln x-\dfrac{1}{x}+c$ $=-\dfrac{1}{x}(\ln x+1)+c$ *	M1 A1 A1 A1 [4]	Integration by parts with $u=\ln x$, du/dx = 1/x, dv/dx = $1/x^2$, $v=-x^{-1}$ must be correct, condone + c condone missing c NB **AG** must have c shown in final answer	Must be correct at this stage . Need to see $1/x^2$

4(i) $h = a - be^{-kt} \Rightarrow a = 10.5$ (their)$a - be^0 = 0.5$ $\Rightarrow b = 10$	B1 M1 A1cao [3]	a need not be substituted	
(ii) $h = 10.5 - 10e^{-kt}$ When $t = 8$, $h = 10.5 - 10e^{-8k} = 6$ $\Rightarrow$ $10e^{-8k} = 4.5$ $\Rightarrow$ $-8k = \ln 0.45$ $\Rightarrow$ $k = \ln 0.45/(-8) = 0.09981\ldots = 0.10$	M1 M1 A1 [3]	ft their a and b (even if made up) taking lns correctly on a correct re-arrangement - ft a, b if not eased cao (www) but allow 0.1	allow M1 for $a - be^{-8k} = 6$ allow a and b unsubstituted allow their 0.45 (or 4.5) to be negative
5 $y = x^2(1 + 4x)^{1/2}$ $\Rightarrow$ $\frac{dy}{dx} = x^2.\frac{1}{2}(1+4x)^{-1/2}.4 + 2x(1+4x)^{1/2}$ $= 2x(1+4x)^{-1/2}(x+1+4x)$ $= \frac{2x(5x+1)}{\sqrt{1+4x}}$ *	M1 B1 A1 M1 A1 [5]	product rule with $u = x^2$, $v = \sqrt{(1 + 4x)}$ ½ $(\ldots)^{-1/2}$ soi correct expression factorising or combining fractions NB **AG**	consistent with their derivatives; condone wrong index in v used for M1 only (need not factor out the $2x$) must have evidence of $x + 1 + 4x$ oe or $2x(5x + 1)(1 + 4x)^{-½}$ or $2x(5x + 1)/(1 + 4x)^{½}$
6(i) $\sin(\pi/3) + \cos(\pi/6) = \sqrt{3}/2 + \sqrt{3}/2 = \sqrt{3}$	B1 [1]	must be exact, must show working	Not just $\sin(\pi/3) + \cos(\pi/6) = \sqrt{3}$, if substituting for y and solving for x (or vv) must evaluate $\sin \pi/3$ e.g. not $\arccos(\sqrt{3} - \sin \pi/3)$
(ii) $2\cos 2x - \sin y\frac{dy}{dx} = 0$ $\Rightarrow$ $2\cos 2x = \sin y\frac{dy}{dx}$ $\Rightarrow$ $\frac{dy}{dx} = \frac{2\cos 2x}{\sin y}$ When $x = \pi/6$, $y = \pi/6$ $\Rightarrow$ $\frac{dy}{dx} = \frac{2\cos \pi/3}{\sin \pi/6} = 2$	M1 A1 A1cao M1dep A1 [5]	Implicit differentiation correct expression substituting dep 1st M1 www	allow one error, but must have (±) $\sin y\, dy/dx$. Ignore $dy/dx = \ldots$ unless pursued. $2\cos 2x\, dx - \sin y\, dy = 0$ is M1A1 (could differentiate wrt y, get dx/dy, etc.) $\frac{-2\cos 2x}{-\sin y}$ is A0 or 30
7 (i) $(3^n + 1)(3^n - 1) = (3^n)^2 - 1$ or $3^{2n} - 1$	B1 [1]	mark final answer	or $9^n - 1$; penalise 3^{n^2} if it looks like 3 to the power n^2.
(ii) 3^n is odd $\Rightarrow 3^n + 1$ and $3^n - 1$ both even As consecutive even nos, one must be divisible by 4, so product is divisible by 8.	M1 M1 A1 [3]	3^n is odd $\Rightarrow 3^n + 1$ and $3^n - 1$ both even completion	Induction: If true for n, $3^{2n} - 1 = 8k$, so $3^{2n} = 1 + 8k$, M1 $3^{2(n+1)} - 1 = 9\times(8k + 1) - 1 = 72k + 8 = 8(9k + 1)$ so div by 8. A1 When $n = 1$, $3^2 - 1 = 8$ div by 8, true A1(or similar with 9^n)

8(i) $f(-x)=\frac{1}{e^{-x}+e^{-(-x)}+2}$ $= f(x)$, [$\Rightarrow$ f is even *] Symmetrical about Oy	M1 A1 B1 [3]	substituting $-x$ for x in f(x) condone 'reflection in y-axis'	Can imply that $e^{-(-x)}=e^x$ from $f(-x)=\frac{1}{e^{-x}+e^x+2}$ Must mention axis
(ii) $f'(x)=-(e^x+e^{-x}+2)^{-2}(e^x-e^{-x})$ or $=\frac{(e^x+e^{-x}+2).0-(e^x-e^{-x})}{(e^x+e^{-x}+2)^2}$ $=\frac{(e^{-x}-e^x)}{(e^x+e^{-x}+2)^2}$	B1 M1 A1 [3]	d/dx (e^x) = e^x and d/dx(e^{-x}) = $-e^{-x}$ soi chain or quotient rule condone missing bracket on top if correct thereafter o.e. mark final answer	. If differentiating $\frac{e^x}{(e^x+1)^2}$ withhold A1 (unless result in (iii) proved here) e.g. $\frac{1}{(e^x+e^{-x}+2)^2}\times(e^{-x}-e^x)$
(iii) $f(x)=\frac{e^x}{e^{2x}+1+2e^x}$ $=\frac{e^x}{(e^x+1)^2}$ *	M1 A1 [2]	× top and bottom by e^x (correctly) condone e^{x^2} for M1 but not A1 **NB AG**	or $\frac{e^x}{(e^x+1)^2}=\frac{e^x}{e^{2x}+2e^x+1}$ M1, $=\frac{1}{e^x+e^{-x}+2}$ A1 condone no $e^{2x}=(e^x)^2$,for both M1 and A1
(iv) $A=\int_0^1\frac{e^x}{(e^x+1)^2}\,dx$ let $u=e^x+1$, $du=e^x\,dx$ when $x=0$, $u=2$; when $x=1$, $u=e+1$ $\Rightarrow$ $A=\int_2^{1+e}\frac{1}{u^2}\,du$ $=\left[-\frac{1}{u}\right]_2^{1+e}$ $=-\frac{1}{1+e}+\frac{1}{2}=\frac{1}{2}-\frac{1}{1+e}$	B1 M1 A1 M1 A1cao [5]	correct integral and limits $\int\frac{1}{u^2}(du)$ $\left[-\frac{1}{u}\right]$ substituting correct limits (dep 1st M1 and integration) o.e. mark final answer. Must be exact Don't allow e^1.	condone no dx, must use $f(x)=\frac{e^x}{(e^x+1)^2}$. Limits may be implied by subsequent work. If 0.231.. unsupported, allow 1st B1 only or by inspection $\left[\frac{k}{e^x+1}\right]$ M1 $\left[-\frac{1}{e^x+1}\right]$ A1 upper–lower; 2 and 1+e (or 3.7..)for u, or 0 and 1 for x if substituted back (correctly) e.g. $\frac{e-1}{2(1+e)}$. Can isw 0.231, which may be used as evidence of M1. Can isw numerical ans (e.g. 0.231) but not algebraic errors
(v) Curves intersect when $f(x)=\frac{1}{4}e^x$ $\Rightarrow$ $(e^x+1)^2=4$ $\Rightarrow$ $e^x=1$ or -3 so as $e^x>0$, only one solution $e^x=1$ $\Rightarrow x=0$ when $x=0$, $y=¼$	M1 M1 A1 B1 B1 [5]	soi or equivalent quadratic – must be correct getting $e^x=1$ and discounting other sol^n $x=0$ www (for this value) $y=¼$ www (for the x value)	$\frac{e^x}{(e^x+1)^2}$ *or* $\frac{1}{e^x+e^{-x}+2}=\frac{1}{4}e^x$ With e^{2x} or $(e^x)^2$, condone e^{x^2}, e^0 www e.g. $e^x=-1$ [or $e^x+1=-2$] not possible www unless verified Do not allow unsupported. A sketch is not sufficient

9(i) When $x = 0$, $f(x) = a = 2$* When $x = \pi$, $f(\pi) = 2 + \sin b\pi = 3$ $\Rightarrow$ $\sin b\pi = 1$ $\Rightarrow$ $b\pi = ½\pi$, so $b = ½$ * *or* $1 = a + \sin(-\pi b)$ $(= a - \sin \pi b)$ $3 = a + \sin(\pi b)$ $\Rightarrow$ $2 = 2\sin \pi b$, $\sin \pi b = 1$, $\pi b = \pi/2$, $b = ½$ $\Rightarrow$ $3 = a + 1$ or $1 = a - 1 \Rightarrow a = 2$ (oe for b)	B1 M1 A1 [3]	NB **AG** 'a is the y-intercept' not enough but allow verification ($2+\sin 0 = 2$) or when $x = -\pi$, $f(-\pi) = 2 + \sin(-b\pi) = 1$ $\Rightarrow \sin(-b\pi) = -1$ condone using degrees $\Rightarrow -b\pi = -½\pi$, $b = ½$ **NB AG** M1 for both points substituted A1 solving for b or a A1 substituting to get a (or b)	or equiv transformation arguments : e.g. 'curve is shifted up 2 so $a = 2$'. e.g. period of sine curve is 4π, or stretched by sf. 2 in x-direction (not squeezed or squashed by ½) $\Rightarrow b = ½$ If verified allow M1A0 If $y = 2 + \sin ½x$ verified at two points, SC2 A sequence of sketches starting from $y = \sin x$ showing clearly the translation and the stretch (in either order) can earn full marks
(ii) $f'(x) = ½\cos ½x$ $\Rightarrow$ $f'(0) = ½$ Maximum value of $\cos ½x$ is 1 $\Rightarrow$ max value of gradient is ½	M1 A1 A1 M1 A1 [5]	$\pm k\cos ½x$ cao www or $f''(x) = -¼\sin ½x$ $f''(x) = 0 \Rightarrow x = 0$, so max val of $f'(x)$ is ½	
(iii) $y = 2 + \sin ½x$ $x \leftrightarrow y$ $x = 2 + \sin ½y$ $\Rightarrow$ $x - 2 = \sin ½y$ $\Rightarrow$ $\arcsin(x - 2) = ½y$ $\Rightarrow$ $y = f^{-1}(x) = 2\arcsin(x - 2)$ Domain $1 \le x \le 3$ Range $-\pi \le y \le \pi$ Gradient at (2, 0) is 2	M1 A1 A1 B1 B1 B1ft [6]	Attempt to invert formula or $\arcsin(y - 2) = ½x$ must be $y = \ldots$ or $f^{-1}(x) = \ldots$ or [1, 3] or $[-\pi, \pi]$ or $-\pi \le f^{-1}(x) \le \pi$ ft their answer in (ii) (except ± 1) 1/their ½	viz solve for x in terms of y or vice-versa – one step enough condone use of a and b in inverse function, e.g. $[\arcsin(x - a)]/b$ or $\sin^{-1}(y - 2)$ condone no bracket for 1st A1 only or $2\sin^{-1}(x - 2)$, condone $f'(x)$, must have bracket in final ans but not $1 \le y \le 3$ but not $-\pi \le x \le \pi$. Penalise <'s (or '1 to 3','$-\pi$ to π') once only or by differentiating $\arcsin(x - 2)$ or implicitly
(iv) $A = \int_0^{\pi}(2 + \sin\frac{1}{2}x)\,dx$ $= \left[2x - 2\cos\frac{1}{2}x\right]_0^{\pi}$ $= 2\pi - (-2)$ $= 2\pi + 2$ $(= 8.2831\ldots)$	M1 M1 A1 A1cao [4]	correct integral and limits $\left[2x - k\cos\frac{1}{2}x\right]$ where k is positive $k = 2$ answers rounding to 8.3	soi from subsequent work, condone no dx but not 180 Unsupported correct answers score 1st M1 only.