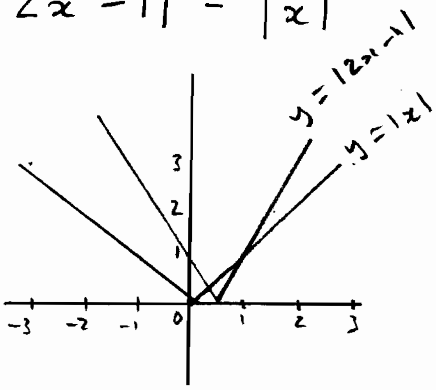
MEI CORE 3 JUNE 2011

1)

$$|2x-1| = |x|$$



Solve $2x - 1 = x$

$$2x - x = 1$$

$$x = 1$$

Solve

$$-2x + 1 = x$$

$$1 = x + 2x$$

$$1 = 3x$$

$$x = \frac{1}{3}$$

Solution $x = \frac{1}{3}, \; x = 1$

2)

$$f(x) = 2\ln x$$

$$g(x) = e^{x}$$

$$gf(x) = g(2\ln x)$$

$$= g(\ln x^2)$$

$$= e^{\ln x^2}$$

$$= x^2$$

3)i $\frac{d}{dx} \frac{\ln x}{x^2}$

$$= \frac{x^2 \frac{1}{x} - 2x\ln x}{x^4}$$

$$= \frac{x(1 - 2\ln x)}{x^4}$$

$$= \frac{1 - 2\ln x}{x^3}$$

3ii) $\int \frac{\ln x}{x^2} \, dx$

Let $u = \ln x$ Let $\frac{dv}{dx} = \frac{1}{x^2}$

$\Rightarrow \frac{du}{dx} = \frac{1}{x}$ $\Rightarrow v = -\frac{1}{x}$

Using $\int u \frac{dv}{dx} = uv - \int v \frac{du}{dx}$

$$\int \frac{\ln x}{x^2} \, dx = -\frac{1}{x}\ln x + \int \frac{1}{x^2} \, dx$$

$$= -\frac{1}{x}\ln x - \frac{1}{x} + c$$

$$= -\frac{1}{x}\left(1 + \ln x\right) + c$$

4) $h = a - be^{-kt}$

i) when $t = \infty, h = 10.5$

$$10.5 = a - be^{-\infty}$$

$$10.5 = a$$

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When $t = 0$, $h = 0.5$

$$0.5 = 10.5 - be^{0}$$

$$b = 10.5 - 0.5 = 10$$

Solution: $a = 10.5$, $b = 10$

4ii) $t = 8$, $h = 6$

$$6 = 10.5 - 10e^{-8k}$$

$$10e^{-8k} = 10.5 - 6$$

$$e^{-8k} = \frac{4.5}{10}$$

$$-8k = \ln 0.45$$

$$k = \frac{\ln 0.45}{-8}$$

$$k = 0.0998$$

$$k = 0.10 \text{ to 2 d.p.}$$

5) $$y = x^2\sqrt{1+4x}$$

$$\frac{dy}{dx} = x^2 \times \tfrac{1}{2}(1+4x)^{-\frac{1}{2}} \times 4 + (1+4x)^{\frac{1}{2}} \times 2x$$

$$= \frac{2x^2}{\sqrt{1+4x}} + 2x\sqrt{1+4x}$$

$$= \frac{2x^2 + 2x(1+4x)}{\sqrt{1+4x}}$$

$$= \frac{10x^2 + 2x}{\sqrt{1+4x}} = \frac{2x(5x+1)}{\sqrt{1+4x}}$$

6)i) $\sin 2x + \cos y = \sqrt{3}$

$P\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$

$$\sin\frac{2\pi}{6} + \cos\frac{\pi}{6}$$

$$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \sqrt{3} \checkmark$$

$\therefore$ on curve

ii) $$2\cos 2x - \sin y \frac{dy}{dx} = 0$$

$$2\cos 2x = \sin y \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2\cos 2x}{\sin y}$$

At $P\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$

$$\frac{dy}{dx} = \frac{2\cos\frac{\pi}{3}}{\sin\frac{\pi}{6}}$$

$$= \frac{2 \times \frac{1}{2}}{\frac{1}{2}} = 2$$

Gradient at $P = 2$

7)i) $$(3^n+1)(3^n-1)$$

$$= 3^{2n} + 3^n - 3^n - 1$$

$$= 3^{2n} - 1$$

7ii) Let n be +ve integer

$$3^{2n-1} = (3^n+1)(3^n-1)$$

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7ii) cont)

If n is a +ve integer

$\Rightarrow 3^n$ is an odd integer

$\Rightarrow 3^n - 1$ and $3^n + 1$ are consecutive even integers

$\Rightarrow$ one is a multiple of 2 and the other a multiple of 4

Their product is therefore a multiple of $2 \times 4 = 8$

$\therefore \quad 3^{2n} - 1$ is divisible by 8

Section B

8) i)

$$f(x) = \frac{1}{e^x + e^{-x} + 2}$$

$$f(-x) = \frac{1}{e^{-x} + e^{x} + 2}$$

$$= \frac{1}{e^x + e^{-x} + 2} = f(x)$$

for all $x \in \mathbb{R}$

$\therefore$ $f(x)$ is an even function and is symmetrical about y-axis

ii) $f(x) = (e^x + e^{-x} + 2)^{-1}$

$$f'(x) = -1(e^x + e^{-x} + 2)^{-2}(e^x - e^{-x})$$

$$= \frac{e^{-x} - e^x}{(e^x + e^{-x} + 2)^2}$$

iii)

$$f(x) = \frac{1}{e^x + e^{-x} + 2}$$

$$= \frac{1}{e^x + e^{-x} + 2} \times \frac{e^x}{e^x}$$

$$= \frac{e^x}{e^{2x} + 1 + 2e^x} = \frac{e^x}{(e^x + 1)^2}$$

iv) Area $= \int_0^1 f(x)\,dx$

$$= \int_0^1 \frac{e^x}{(e^x+1)^2}\,dx$$

Let $u = e^x + 1$

$$\Rightarrow \frac{du}{dx} = e^x$$

$$\Rightarrow du = e^x\,dx$$

when $x = 1 \quad u = 1 + e$

when $x = 0 \quad u = 2$

$$\int_0^1 f(x)\,dx = \int_2^{1+e} \frac{1}{u^2}\,du$$

$$= \left[-\frac{1}{u}\right]_2^{1+e}$$

$$= -\frac{1}{1+e} + \frac{1}{2}$$

v)

$y = f(x) \quad (1)$

$y = \frac{1}{4}e^x \quad (2)$

Sub for y in (1)

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8v cont)

$$\frac{1}{4}e^x = \frac{e^x}{(e^x+1)^2}$$

$$e^x = \frac{4e^x}{(e^x+1)^2}$$

$$(e^x+1)^2 e^x = 4e^x$$

$$(e^x+1)^2 e^x - 4e^x = 0$$

$$e^x\left[(e^x+1)^2 - 4\right] = 0$$

$$\Rightarrow (e^x+1)^2 = 4$$

$$\Rightarrow e^x + 1 = \pm 2$$

When $e^x + 1 = -2$

$e^x = -3$ ✗ no solution

so only solution when

$$e^x + 1 = 2$$

$$e^x = 1$$

$$\Rightarrow x = 0$$

when $x = 0$ $y = \frac{1}{4}e^0 = \frac{1}{4}$

$(0, \frac{1}{4})$ is the only point of intersection.

9) $f(x) = a + \sin bx$

i) $P(-\pi, 1)$ $Q(\pi, 3)$

From graph when $x = 0$, $f(x) = 2$

so $2 = a + \sin 0$

$$\Rightarrow a = 2$$

Curve passes through $(\pi, 3)$

$$3 = 2 + \sin b\pi$$

$$1 = \sin b\pi$$

$$\Rightarrow b = \frac{1}{2}$$

ii) $f(x) = 2 + \sin\frac{x}{2}$

$$\Rightarrow f'(x) = \frac{1}{2}\cos\frac{x}{2}$$

At $(0, 2)$,

$$\frac{dy}{dx} = \frac{1}{2}\cos 0 = \frac{1}{2}$$

Since $-1 \le \cos\frac{x}{2} \le 1$

$$-\frac{1}{2} \le \frac{1}{2}\cos\frac{x}{2} \le \frac{1}{2}$$

$$\therefore -\frac{1}{2} \le f'(x) \le \frac{1}{2}$$

so no point on curve with gradient $> \frac{1}{2}$

iii) Let $y = 2 + \sin\frac{x}{2}$

Swap variables and rearrange

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9 iii) cont

$$x = 2 + \sin\frac{y}{2}$$

$$x - 2 = \sin\frac{y}{2}$$

$$\sin^{-1}(x-2) = \frac{y}{2}$$

$$y = 2\sin^{-1}(x-2)$$

$$\therefore \quad f^{-1}(x) = 2\sin^{-1}(x-2)$$

Its domain is the range of $f(x)$

domain $1 \le x \le 3$

Its range is the domain of $f(x)$

Range $-\pi \le f^{-1}(x) \le \pi$

Gradient of $f(x)$ at $(0,2)$ was found to be $\frac{1}{2}$

$\therefore$ Gradient of $f^{-1}(x)$ at corresponding point $(2,0)$ is given by

$$\frac{1}{\frac{1}{2}} = 2$$

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9 iv)

$$\text{Area} = \int_0^{\pi} f(x)\,dx$$

$$= \int_0^{\pi} \left(2 + \sin\frac{x}{2}\right) dx$$

$$= \left[2x - 2\cos\frac{x}{2}\right]_0^{\pi}$$

$$= \left(2\pi - 2\cos\frac{\pi}{2}\right) - \left(0 - 2\cos 0\right)$$

$$= 2\pi + 2 \qquad \text{units}^2$$