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1)
$$\frac{d}{dx} x^2 \tan 2x$$
$$= x^2 \times 2\sec^2 2x + \tan 2x \times 2x$$
$$= 2x^2 \sec^2 2x + 2x \tan 2x$$

2)
$$f(x) = \ln x$$
$$g(x) = 1 + x^2$$

$$fg(x) = f(1+x^2)$$
$$= \ln(1+x^2)$$

which is an even function

$$gf(x) = g(\ln x)$$
$$= 1 + (\ln x)^2$$

which is neither odd nor even

(not defined for $x \le 0$)

3)
$$\int_0^{\frac{\pi}{2}} x \cos\frac{x}{2}\, dx$$

Using $\int u \frac{dv}{dx} = uv - \int v \frac{du}{dx}$

Let $u = x$ $\Rightarrow \frac{du}{dx} = 1$

Let $\frac{dv}{dx} = \cos\frac{x}{2}$ $\Rightarrow v = 2\sin\frac{x}{2}$

$$\int_0^{\frac{\pi}{2}} x \cos\frac{x}{2}\, dx$$
$$= \left[2x \sin\frac{x}{2}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2\sin\frac{x}{2}\, dx$$
$$= \left[2x \sin\frac{x}{2}\right]_0^{\frac{\pi}{2}} - \left[-4\cos\frac{x}{2}\right]_0^{\frac{\pi}{2}}$$
$$= \left[\pi \sin\frac{\pi}{4} - 0\right] + 4\left[\cos\frac{\pi}{4} - \cos 0\right]$$
$$= \frac{\pi}{\sqrt{2}} + \frac{4}{\sqrt{2}} - 4$$
$$= \frac{\sqrt{2}}{2}\pi + 2\sqrt{2} - 4$$

4) 'No cube of an integer has 2 as its digit unit'

$0^3 = 0$, $1^3 = 1$, $2^3 = 8$,
$3^3 = 27$, $4^3 = 64$, $5^3 = 125$
$6^3 = 216$, $7^3 = 343$, $8^3 = 512$

Statement is false since $8^3 = 512$ which has 2 as its digit unit.

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5 i) $y = |x|$

Tranlate by $\begin{pmatrix}1\\0\end{pmatrix}$ then one-way stretch parallel to y-axis scale factor 2.

$$f(x) = 2|x-1|$$

ii) $y = \cos x$

Reflect in x-axis then translate by $\begin{pmatrix}0\\1\end{pmatrix}$

$$g(x) = 1 - \cos x$$

6) $$r = 20(1 - e^{-0.2t})$$

i) When $t = 2$,

$$r = 20(1 - e^{-0.4})$$

$$r = 6.59 \text{ m} \quad \text{to 3 s.f.}$$

$$\frac{dr}{dt} = 20 \times 0.2e^{-0.2t}$$

$$= 4e^{-0.2t}$$

when $t = 2$,

$$\frac{dr}{dt} = 4e^{-0.4} = 2.68 \text{ ms}^{-1}$$

to 3 s.f.

ii) $$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

Chain Rule $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$$\frac{dA}{dt} = 2\pi r \times 4e^{-0.2t}$$

when $t = 2$

$$\frac{dA}{dt} = 2\pi \times 20(1 - e^{-0.4}) \times 4e^{-0.4}$$

$$\frac{dA}{dt} = 111.08 \text{ m}^2\text{s}^{-1}$$

$$= 111 \text{ m}^2\text{s}^{-1} \quad \text{to 3 s.f.}$$

Note $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$$= 2\pi \times 6.59 \times 2.68$$

$$= 110.96859$$

$$= 111 \text{ m}^2\text{s}^{-1} \quad \text{to 3 s.f.}$$

so using previous rounded answers would still give 3 s.f. accuracy. However, it is usually safer to use unrounded answers in further calculations

7i) $$x^3 + y^3 = 3xy$$

Implicit differentiation

$$3x^2 + 3y^2\frac{dy}{dx} = 3x\frac{dy}{dx} + 3y$$

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7i cont) Divide throughout by 3

$$x^2 + y^2\frac{dy}{dx} = x\frac{dy}{dx} + y$$

$$y^2\frac{dy}{dx} - x\frac{dy}{dx} = y - x^2$$

$$\frac{dy}{dx}(y^2 - x) = y - x^2$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

7ii) At turning point P, $\frac{dy}{dx} = 0$

$$\Rightarrow \quad y - x^2 = 0$$

$$\Rightarrow \quad y = x^2$$

Sub for y in

$$x^3 + y^3 = 3xy$$

$$x^3 + x^6 = 3x \times x^2$$

$$x^3 + x^6 = 3x^3$$

$$x^6 = 2x^3$$

$$x^3 = 2$$

$$x = 2^{\frac{1}{3}}$$

Section B

8i)

$$y = \frac{x}{\sqrt{x-2}}$$

When $x = 3$, $y = \frac{3}{\sqrt{3-2}} = 3$

$\therefore$ (3, 3) is on curve

so P is point (3, 3)

8ii)

$$\frac{d}{dx}\frac{u}{v} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{\sqrt{x-2} \times 1 - x \times \frac{1}{2}(x-2)^{-\frac{1}{2}}}{\sqrt{x-2}^{\,2}}$$

$$\frac{dy}{dx} = \frac{\sqrt{x-2} - \frac{x}{2\sqrt{x-2}}}{\sqrt{x-2}^{\,2}}$$

Multiply numerator and denominator by $\sqrt{x-2}$

$$\frac{dy}{dx} = \frac{x - 2 - \frac{x}{2}}{(x-2)^{3/2}}$$

$$\frac{dy}{dx} = \frac{\frac{2x - 4 - x}{2}}{(x-2)^{3/2}}$$

$$\frac{dy}{dx} = \frac{x - 4}{2(x-2)^{3/2}}$$

At P(3, 3)

$$\frac{dy}{dx} = \frac{3 - 4}{2(3-2)^{3/2}} = -\frac{1}{2}$$

Not symmetrical about $y = x$

since $\frac{dy}{dx} \neq -1$

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8iii) $\int_3^{11} \frac{x}{\sqrt{x-2}}\,dx$

Let $u = x-2$

$\Rightarrow \frac{du}{dx} = 1$

$\Rightarrow du = dx$

when $x = 11$, $u = 9$

when $x = 3$, $u = 1$

Also note that $x = u+2$

$\therefore \int_3^{11} \frac{x}{\sqrt{x-2}}\,dx = \int_1^9 \frac{u+2}{\sqrt{u}}\,du$

$= \int_1^9 \left(u^{\frac{1}{2}} + 2u^{-\frac{1}{2}}\right) du$

$= \left[\frac{u^{3/2}}{3/2} + \frac{2u^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^9$

$= \left[\frac{2}{3}u^{3/2} + 4u^{1/2}\right]_1^9$

$= \left(\frac{2}{3} \times 27 + 4 \times 3\right) - \left(\frac{2}{3} + 4\right)$

$= 18 + 12 - 4\frac{2}{3}$

$= 25\frac{1}{3}$

Area of region PQR

= Area under $y = x$ between $x = 3$ and $x = 11$ minus area under curve $\int_3^{11} \frac{x}{\sqrt{x-2}}\,dx$

$= (3+11) \times \frac{8}{2} - 25\frac{1}{3}$

$= 56 - 25\frac{1}{3} = 30\frac{2}{3}$ units^2

(Note area under $y = x$ is a trapezium)

9i) $f(x) = \ln\left(\frac{2x}{1+x}\right), x > 0$

At P, $y = f(x) = 0$

$\Rightarrow \ln\left(\frac{2x}{1+x}\right) = 0$

$\Rightarrow \frac{2x}{1+x} = 1$

$\Rightarrow 2x = 1+x$

$\Rightarrow x = 1$

$\therefore P(1, 0)$

When $x = 2$

$y = f(x) = \ln\left(\frac{2 \times 2}{1+2}\right) = \ln\left(\frac{4}{3}\right)$

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9i cont) $\therefore \quad Q\left(2, \ln\left(\frac{4}{3}\right)\right)$

9ii) $f(x) = \ln\left(\frac{2x}{1+x}\right)$

$f(x) = \ln(2x) - \ln(1+x)$

$f'(x) = \frac{2}{2x} - \frac{1}{1+x}$

$f'(x) = \frac{1}{x} - \frac{1}{1+x}$

At $P(1,0)$

$f'(1) = \frac{1}{1} - \frac{1}{1+1} = \frac{1}{2}$

Gradient at $P = \frac{1}{2}$

9iii) $g(x) = \frac{e^x}{2-e^x}, \; x < \ln 2$

$f(x) = y = \ln\left(\frac{2x}{1+x}\right)$

Swap variables and rearrange

$x = \ln\left(\frac{2y}{1+y}\right)$

$e^x = \frac{2y}{1+y}$

$(1+y)e^x = 2y$

$e^x + ye^x = 2y$

$e^x = 2y - ye^x$

$e^x = y(2-e^x)$

$\frac{e^x}{2-e^x} = y$

$\therefore \quad f^{-1}(x) = \frac{e^x}{2-e^x}$

$\therefore \quad f^{-1}(x) = g(x)$

Gradient of $g(x)$ at R is the positive reciprocal of gradient of $f(x)$ at corresponding point P

Gradient of $g(x)$ at $R = 2$

9iv) $\int_0^{\ln\frac{4}{3}} g(x)\,dx$

$= \int_0^{\ln\frac{4}{3}} \frac{e^x}{2-e^x}\,dx$

Let $u = 2 - e^x$

$\Rightarrow \frac{du}{dx} = -e^x$

$\Rightarrow du = -e^x dx$

$\Rightarrow -du = e^x dx$

When $x = \ln\frac{4}{3}$, $u = 2 - \frac{4}{3} = \frac{2}{3}$

When $x = 0$, $u = 2 - 1 = 1$

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9 iv cont)

$$\int_0^{\ln\frac{4}{3}} \frac{e^x}{2-e^x}\,dx$$

$$= \int_1^{2/3} -\frac{1}{u}\,du$$

$$= \int_{2/3}^{1} \frac{1}{u}\,du$$

$$= \Big[\ln u\Big]_{2/3}^{1}$$

$$= \ln 1 - \ln\frac{2}{3}$$

$$= 0 - \ln\frac{2}{3}$$

$$= \ln\frac{3}{2}$$

Alternatively, because in $\frac{e^x}{2-e^x}$ the numerator is minus the differential of the denominator, we can deduce that the integral is $-\ln(\text{denominator})$

$$\int_0^{\ln\frac{4}{3}} \frac{e^x}{2-e^x}\,dx$$

$$= \Big[-\ln(2-e^x)\Big]_0^{\ln\frac{4}{3}}$$

$$= -\ln\left(2-\frac{4}{3}\right) - -\ln(2-1)$$

$$= -\ln\frac{2}{3} + \ln 1$$

$$= \ln\frac{3}{2}$$

Area shaded given by

$$\int_1^2 f(x)\,dx$$

$$= 2\times\ln\frac{4}{3} - \int_0^{\ln\frac{4}{3}} g(x)\,dx$$

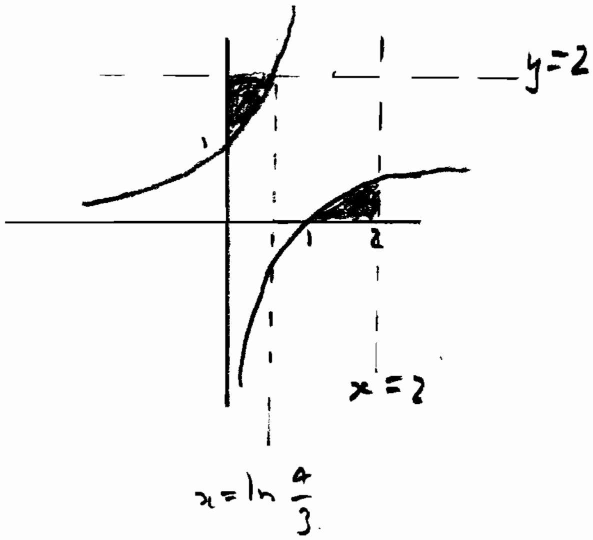
$$= 2\ln\frac{4}{3} - \ln\frac{3}{2}$$

$$= \ln\left(\frac{4}{3}\right)^2 - \ln\frac{3}{2}$$

$$= \ln\frac{16}{9} - \ln\frac{3}{2}$$

$$= \ln\left(\frac{\frac{16}{9}}{\frac{3}{2}}\right) = \ln\left(\frac{16}{9}\times\frac{2}{3}\right)$$

$$= \ln\left(\frac{32}{27}\right)$$



This is rectangle − area under g(x)

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