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1) $$\int_1^2 \frac{1}{\sqrt{3x-2}}\,dx$$

Let $u = 3x-2$

$$\frac{du}{dx} = 3$$

$$du = 3dx$$

$$\tfrac{1}{3}du = dx$$

When $x = 1$, $u = 3(1)-2 = 1$

when $x = 2$, $u = 3(2)-2 = 4$

$$\int_1^4 \frac{1}{\sqrt{u}}\,\frac{1}{3}\,du$$

$$= \int_1^4 \tfrac{1}{3}u^{-\frac{1}{2}}\,du$$

$$= \frac{1}{3}\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^4$$

$$= \frac{1}{3}\left[2\sqrt{u}\right]_1^4$$

$$= \frac{1}{3}\left[4-2\right] = \frac{2}{3}$$

2) $$|2x+1| > 4$$

Either $2x+1 > 4$

$2x > 4-1$

$2x > 3$

$x > \frac{3}{2}$

or $2x+1 < -4$

$2x < -4-1$

$2x < -5$

$x < -\frac{5}{2}$

Solution Either $x > \frac{3}{2}$

or $x < -\frac{5}{2}$

3) $$e^{2y} = 5 - e^{-x}$$

$$2e^{2y}\frac{dy}{dx} = e^{-x}$$

$$\frac{dy}{dx} = \frac{e^{-x}}{2e^{2y}}$$

At $(0, \ln 2)$

$$\frac{dy}{dx} = \frac{e^{0}}{2e^{2\ln 2}} = \frac{1}{2e^{\ln 4}}$$

$$\frac{dy}{dx} = \frac{1}{2\times 4} = \frac{1}{8}$$

4) i) $$f(x) = \sqrt{1-9x^2}$$

$-a \le x \le a$

$$0 = \sqrt{1-9x^2}$$

$$0 = 1-9x^2$$

$$9x^2 = 1$$

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4 cont) $x^2 = \frac{1}{9}$

$x = \pm\frac{1}{3}$

so $a = \frac{1}{3}$

ii) When $x = 0$ $f(x) = \sqrt{1-0} = 1$

Range $0 \le f(x) \le 1$

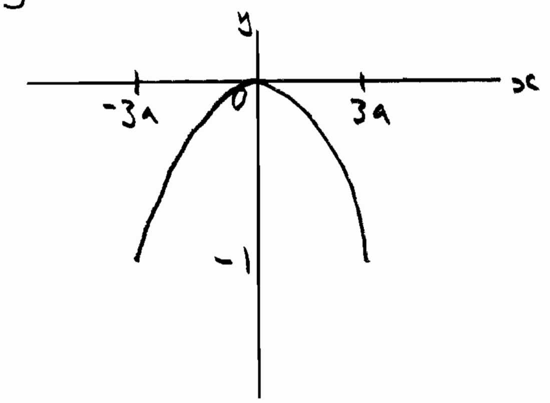
iii) $y = f\left(\frac{1}{3}x\right) - 1$

Transformations of original graph

One way stretch parallel to x-axis scale factor 3

Translation by $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$y = f\left(\frac{1}{3}x\right) - 1$



5) $P = 7 - 2e^{-kt}$

i) When $t = 0$, $P = 7 - 2e^0$

$P = 5$

At $t = 0$, population = 5 million

As $t \to \infty$, $P \to 7 - 2e^{-\infty}$

$P \to 7$

long term population → 7 million

ii) When $t = 1$, $P = 5.5$

$5.5 = 7 - 2e^{-k\times 1}$

$2e^{-k} = 7 - 5.5$

$e^{-k} = \frac{1.5}{2} = 0.75$

$-k = \ln 0.75$

$k = -\ln 0.75$

$k = 0.288$ to 3 s.f.

6) $y = f(x) = 2\sin^{-1}x$

i) $-1 \le x \le 1$

At P, $x = \frac{1}{2}$

$y = 2\sin^{-1}\frac{1}{2} = 2 \times \frac{\pi}{6} = \frac{\pi}{3}$

$y = \frac{\pi}{3}$

ii) Swap variables and rearrange

$x = 2\sin^{-1}y$

$\frac{x}{2} = \sin^{-1}y$

$\sin\left(\frac{x}{2}\right) = y$

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6ii cont)

$\therefore g(x) = \sin\left(\frac{x}{2}\right)$

$g'(x) = \frac{1}{2}\cos\left(\frac{x}{2}\right)$

$P\left(\frac{1}{2}, \frac{\pi}{3}\right)$ so $Q\left(\frac{\pi}{3}, \frac{1}{2}\right)$

At Q $g'\left(\frac{\pi}{3}\right) = \frac{1}{2}\cos\left(\frac{\pi}{6}\right)$

$= \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$

$\therefore$ gradient of $y = g(x)$ at Q is $\frac{\sqrt{3}}{4}$

Gradient of $y = f(x)$ at P will be $\frac{4}{\sqrt{3}}$

7) i) $f(x)$ and $g(x)$ odd functions

$\Rightarrow f(-x) = -f(x)$

and $g(-x) = -g(x)$

for $x \in \mathbb{R}$

$s(x) = f(x) + g(x)$

$s(-x) = f(-x) + g(-x)$

$s(-x) = -f(x) - g(x)$

$s(-x) = -\left(f(x) + g(x)\right)$

$s(-x) = -s(x)$

$\therefore s(x)$ is also an odd function

7ii) $p(x) = f(x)g(x)$

$p(-x) = f(-x)g(-x)$

$p(-x) = -f(x)\left(-g(x)\right)$

$p(-x) = f(x)g(x)$

$p(-x) = p(x)$ for $x \in \mathbb{R}$

$\therefore p(x)$ is an even function

Section B

8) $y = x\sin 2x$

i)

$\frac{dy}{dx} = x \times 2\cos 2x + \sin 2x$

$\frac{dy}{dx} = 2x\cos 2x + \sin 2x$

At turning points $\frac{dy}{dx} = 0$

$\Rightarrow 2x\cos 2x + \sin 2x = 0$

$\div \cos 2x$

$2x + \tan 2x = 0$

ii) At P, $y = 0$

$\Rightarrow 0 = x\sin 2x$

$\Rightarrow x = 0$ or $\sin 2x = 0$

$2x = 0, \pi, 2\pi$

$x = 0, \frac{\pi}{2}, \pi$

From graph $x = \frac{\pi}{2}$

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8ii) cont)

$$\frac{dy}{dx} = 2x\cos 2x + \sin 2x$$

At P where $x = \frac{\pi}{2}$

$$\frac{dy}{dx} = \pi\cos\pi + \sin\pi$$

$$= \pi \times -1 + 0$$

$$\frac{dy}{dx} = -\pi$$

Tgt gradient $-\pi$ thro $(\frac{\pi}{2}, 0)$

Using $y - y_1 = m(x - x_1)$

$$y - 0 = -\pi(x - \tfrac{\pi}{2})$$

$$y = -\pi x + \frac{\pi^2}{2}$$

$$2y = -\pi x + \pi^2$$

$$2\pi x + 2y = \pi^2$$

Q is on tgt with $x = 0$

$$0 + 2y = \pi^2$$

$$y = \frac{\pi^2}{2}$$

so $Q\left(0, \frac{\pi^2}{2}\right)$

8iii) Area shaded is given by Area under tangent − Area under curve

Area under tgt $= \frac{1}{2} \times \frac{\pi}{2} \times \frac{\pi^2}{2}$

$$= \frac{\pi^3}{8}$$

Area under curve

$$= \int_0^{\frac{\pi}{2}} x\sin 2x \, dx$$

use $\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$

Let $u = x$ $\Rightarrow \frac{du}{dx} = 1$

Let $\frac{dv}{dx} = \sin 2x$ $\Rightarrow v = -\frac{1}{2}\cos 2x$

$$\int_0^{\frac{\pi}{2}} x\sin 2x\,dx = \left[-\tfrac{1}{2}x\cos 2x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \tfrac{1}{2}\cos 2x\,dx$$

$$= \left[-\tfrac{1}{2}x\cos 2x\right]_0^{\frac{\pi}{2}} + \left[\tfrac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}}$$

$$= -\frac{\pi}{4}\cos\pi - 0 + \sin\pi - 0$$

$$= \frac{\pi}{4}$$

$\therefore$ shaded area $= \frac{\pi^3}{8} - \frac{\pi}{4}$

$$= \frac{\pi}{8}\left[\pi^2 - 2\right]$$

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9) i) $y = f(x)$ $P(0,3)$ $Q(1,2)$

asymptote $x = -1$

A) $y = 2f(x)$

This represents a one-way stretch parallel to y-axis scale factor 2

$P(0,3) \rightarrow (0,6)$

$Q(1,2) \rightarrow (1,4)$

B) $y = f(x+1) + 2$

This represents a translation by $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$P(0,3) \rightarrow (-1,5)$

$Q(1,2) \rightarrow (0,4)$

ii) $f(x) = \dfrac{x^2+3}{x+1}$ $x \neq 1$

$$\frac{d}{dx}\frac{u}{v} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$f'(x) = \frac{(x+1) \times 2x - (x^2+3) \times 1}{(x+1)^2}$$

$$f'(x) = \frac{2x^2+2x-x^2-3}{(x+1)^2}$$

$$f'(x) = \frac{x^2+2x-3}{(x+1)^2}$$

$$f'(x) = \frac{(x+3)(x-1)}{(x+1)^2}$$

At t.p. $\dfrac{dy}{dx} = 0$

$\Rightarrow (x+3)(x-1) = 0$

$\Rightarrow x = -3$ or $x = 1$

When $x = -3$, $y = \dfrac{(-3)^2+3}{-3+1}$

$y = \dfrac{12}{-2} = -6$

Other turning point is $(-3, -6)$

9 iii) $f(x-1) = \dfrac{(x-1)^2+3}{(x-1)+1}$

$$= \frac{x^2-2x+1+3}{x}$$

$$= x - 2 + \frac{4}{x}$$

9 iv) $\displaystyle\int_a^b \left(x - 2 + \frac{4}{x}\right) dx$

$$= \left[\frac{x^2}{2} - 2x + 4\ln x\right]_a^b$$

$$= \frac{b^2}{2} - 2b + 4\ln b - \frac{a^2}{2} + 2a - 4\ln a$$

$y = f(x)$ has been translated by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

so choose $a = 1$, $b = 2$

$$\text{Area} = \frac{2^2}{2} - 2(2) + 4\ln 2 - \frac{1^2}{2} + 2(1) - 4\ln 1$$

$$= 2 - 4 + 4\ln 2 - \frac{1}{2} + 2 - 0$$

$$= 4\ln 2 - \frac{1}{2}$$