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1) i)

$$y = a|x+b|$$

$(-1, 0)$ on graph

$$0 = a|-1+b|$$

$$\Rightarrow \underline{b = 1}$$

so $y = a|x+1|$

$(0, \frac{1}{2})$ on graph

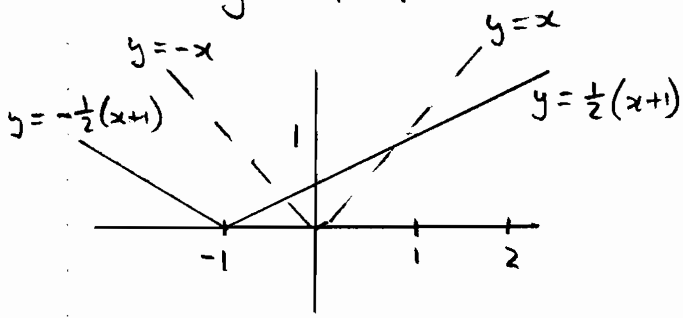
$$\frac{1}{2} = a|0+1|$$

$$\Rightarrow \underline{a = \frac{1}{2}}$$

ii)

$$y = \frac{1}{2}|x+1|$$

$$y = |x|$$



Solve $y = x$ (1)

$y = \frac{1}{2}(x+1)$ (2)

Sub for y in (2)

$$x = \frac{1}{2}x + \frac{1}{2}$$

$$x - \frac{1}{2}x = \frac{1}{2}$$

$$\frac{1}{2}x = \frac{1}{2}$$

$$x = 1 \quad \Rightarrow y = 1$$

Intersection at $(1, 1)$

Solve $y = -x$ (3)

$y = \frac{1}{2}(x+1)$ (4)

Sub for y in (4)

$$-x = \frac{1}{2}x + \frac{1}{2}$$

$$-x - \frac{1}{2}x = \frac{1}{2}$$

$$-\frac{3}{2}x = \frac{1}{2}$$

$$x = \frac{1}{2} \times -\frac{2}{3}$$

$$x = -\frac{1}{3}$$

$$\Rightarrow y = \frac{1}{3}$$

Intersection at $(-\frac{1}{3}, \frac{1}{3})$

2) $n^3 - n = n(n^2-1)$

i) $= n(n+1)(n-1)$

$= (n-1)n(n+1)$

ii) If n an integer

$(n-1), n, (n+1)$ are consecutive integers. One must be a multiple of 3. At least one must be a multiple of 2

$\therefore$ 2 and 3 are both factors of $(n-1)n(n+1)$

Since they are coprime $2 \times 3 = 6$ is also a factor

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$\therefore\ n^3 - n = (n-1)n(n+1)$

is divisible by 6

3) $f(x) = 1 - 2\sin x$

$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

i) Range $-1 \le f(x) \le 3$

ii) Let $y = 1 - 2\sin x$

Swap variables and rearrange

$x = 1 - 2\sin y$

$2\sin y = 1 - x$

$\sin y = \frac{1-x}{2}$

$y = \sin^{-1}\left(\frac{1-x}{2}\right)$

so $f^{-1}(x) = \sin^{-1}\left(\frac{1-x}{2}\right)$

iii) $f'(x) = -2\cos x$

$f'(0) = -2\cos 0 = -2$

Gradient at corresponding point on $y = f^{-1}(x)$ ie at $(1,0)$

$= -\frac{1}{2}$

4) $V = \pi h^2$

$\Rightarrow \frac{dV}{dh} = 2\pi h$

Given $\frac{dV}{dt} = 10\ cm^3 s^{-1}$

Find $\frac{dh}{dt}$ when $h = 5cm$

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$= \frac{1}{\frac{dV}{dh}} \times \frac{dV}{dt}$

$= \frac{1}{2\pi h} \times \frac{dV}{dt}$

when $h = 5cm$

$\frac{dh}{dt} = \frac{1}{2 \times \pi \times 5} \times 10 \quad cm\,s^{-1}$

$= \frac{1}{\pi}\ cm\,s^{-1}$

or $0.3183\ cm\,s^{-1}$

5) $y = \ln\left(\sqrt{\frac{2x-1}{2x+1}}\right)$

$y = \ln(2x-1)^{\frac{1}{2}} - \ln(2x+1)^{\frac{1}{2}}$

$y = \frac{1}{2}\ln(2x-1) - \frac{1}{2}\ln(2x+1)$

$\frac{dy}{dx} = \frac{1}{2} \times \frac{2}{(2x-1)} - \frac{1}{2} \times \frac{2}{(2x+1)}$

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5 cont) $\frac{dy}{dx} = \frac{1}{2x-1} - \frac{1}{2x+1}$

6) $\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{3+\cos 2x}\,dx$

$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{-2\sin 2x}{3+\cos 2x}\,dx$

$= -\frac{1}{2}\left[\ln(3+\cos 2x)\right]_0^{\frac{\pi}{2}}$

$= -\frac{1}{2}\left[\ln 2 - \ln 4\right]$

$= \frac{1}{2}\left[\ln 4 - \ln 2\right]$

$= \frac{1}{2}\ln\frac{4}{2} = \frac{1}{2}\ln 2$

6) Alternative method using a substitution

$\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{3+\cos 2x}\,dx$

Let $u = 3 + \cos 2x$

$\frac{du}{dx} = -2\sin 2x$

$du = -2\sin 2x\,dx$

$-\frac{1}{2}du = \sin 2x\,dx$

when $x = 0$, $u = 3+1 = 4$

when $x = \frac{\pi}{2}$, $u = 3-1 = 2$

$\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{3+\cos 2x}\,dx$

$= \int_4^2 \frac{1}{u} \times \frac{-1}{2}\,du$

$= -\frac{1}{2}\int_4^2 \frac{1}{u}\,du$

$= -\frac{1}{2}\left[\ln u\right]_4^2$

$= -\frac{1}{2}\left[\ln 2 - \ln 4\right]$

$= \frac{1}{2}\left[\ln 4 - \ln 2\right]$

$= \frac{1}{2}\ln\frac{4}{2} = \frac{1}{2}\ln 2$

7) i) $f(x) = \frac{2x}{1-x^2}$

$f(-x) = \frac{2(-x)}{1-(-x)^2} = \frac{-2x}{1-x^2}$

$= -\frac{2x}{1-x^2}$

$= -f(x)$

$\therefore f(x)$ is an odd function

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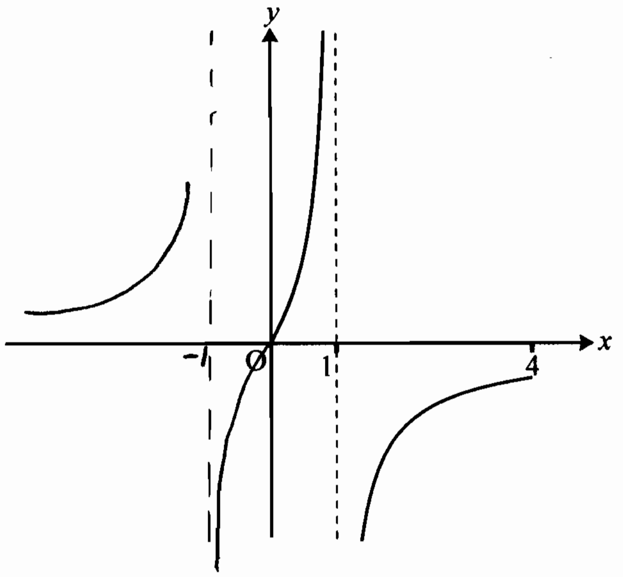
7ii)



$\Rightarrow e^{2x}(1-2x) = 0$

$\Rightarrow 1-2x = 0$

$\Rightarrow x = \frac{1}{2}$

When $x = \frac{1}{2}$,

$y = (1-\frac{1}{2})e^{2\times\frac{1}{2}}$

$y = \frac{1}{2}e$

$\therefore P(\frac{1}{2}, \frac{e}{2})$

Section B

8) $y = (1-x)e^{2x}$

i) when $x=0$, $y = 1\times e^0 = 1$

when $y=0$, $1-x=0$
$x = 1$

Intercept with y-axis $(0,1)$

Intercept with x-axis $(1,0)$

ii) $\frac{dy}{dx} = (1-x)\times 2e^{2x} + e^{2x}(-1)$

$= 2e^{2x} - 2xe^{2x} - e^{2x}$

$= e^{2x} - 2xe^{2x}$

$= e^{2x}(1-2x)$

At t.p. $\frac{dy}{dx} = 0$

iii) Area $= \int_0^1 e^{2x}(1-x)\,dx$

Using $\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$

Let $u = 1-x$ Let $\frac{dv}{dx} = e^{2x}$

$\Rightarrow \frac{du}{dx} = -1$ $\Rightarrow v = \frac{1}{2}e^{2x}$

$\int_0^1 e^{2x}(1-x)\,dx$

$= \left[\frac{1}{2}e^{2x}(1-x)\right]_0^1 + \int_0^1 \frac{1}{2}e^{2x}\,dx$

$= \left[\frac{1}{2}e^{2x}(1-x)\right]_0^1 + \left[\frac{1}{4}e^{2x}\right]_0^1$

$= \left[\frac{3}{4}e^{2x} - \frac{1}{2}xe^{2x}\right]_0^1$

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8 iii cont)

$$= \left(\tfrac{3}{4}e^2 - \tfrac{1}{2}e^2\right) - \left(\tfrac{3}{4} - 0\right)$$

$$= \tfrac{1}{4}e^2 - \tfrac{3}{4}$$

$$= \tfrac{1}{4}\left(e^2 - 3\right)$$

v)

New area = original area × 3 × 2

$$= \tfrac{1}{4}\left(e^2-3\right) \times 6$$

$$= \tfrac{3}{2}\left(e^2-3\right)$$

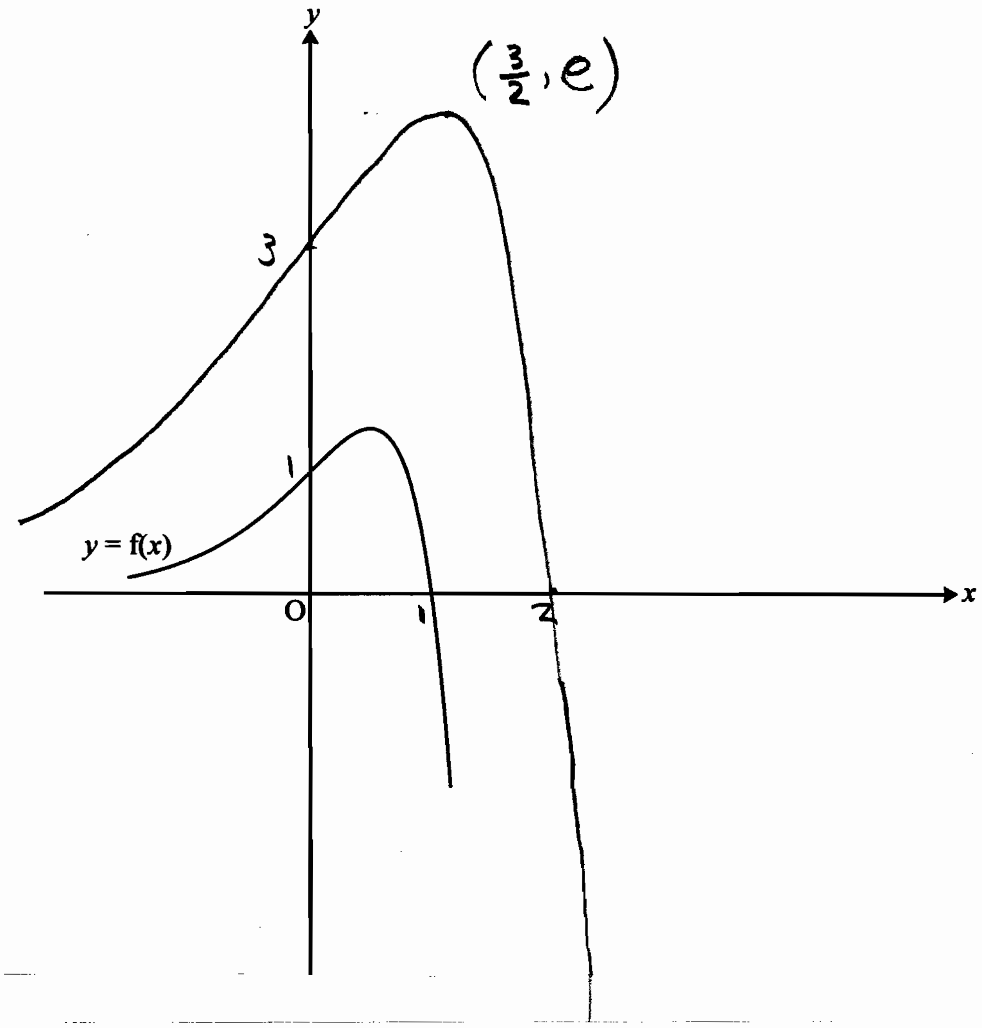
8 iv)

$$g(x) = 3f\left(\tfrac{1}{2}x\right)$$

$$g(x) = 3\left(\left(1-\tfrac{x}{2}\right)e^x\right)$$

$$g(x) = 3\left(1-\tfrac{x}{2}\right)e^x$$

This represents a two way stretch. s.f. 3 parallel to y-axis and s.f. 2 parallel to x-axis.



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9) $y^3 = \dfrac{x^3}{2x-1}$

i) $a = \frac{1}{2}$

ii) $3y^2\dfrac{dy}{dx} = \dfrac{(2x-1)3x^2 - x^3 \times 2}{(2x-1)^2}$

$3y^2\dfrac{dy}{dx} = \dfrac{6x^3 - 3x^2 - 2x^3}{(2x-1)^2}$

$\dfrac{dy}{dx} = \dfrac{4x^3 - 3x^2}{3y^2(2x-1)^2}$

At t.p. $\dfrac{dy}{dx} = 0$

$\Rightarrow 4x^3 - 3x^2 = 0$

$\Rightarrow x^2(4x-3) = 0$

$\Rightarrow x = 0$ or $x = \frac{3}{4}$

At $x=0$, $y=0$ and $\dfrac{dy}{dx} \neq 0$

At P, $x = \frac{3}{4}$

$y^3 = \dfrac{0.75^3}{1.5-1} = \dfrac{0.75^3}{0.5}$

$y = \sqrt[3]{\dfrac{0.75^3}{0.5}}$

$y = 0.94494$

$y = 0.945$ (to 3 s.f.)

$P(0.75, 0.945)$

9 iii) $\displaystyle\int \frac{x}{\sqrt[3]{2x-1}}\,dx$

Let $u = 2x-1 \Rightarrow x = \dfrac{u+1}{2}$

$\dfrac{du}{dx} = 2$

$du = 2\,dx$

$\frac{1}{2}du = dx$

Integral becomes

$\displaystyle\int \frac{\frac{u+1}{2}}{u^{1/3}}\frac{1}{2}du$

$= \displaystyle\int \frac{u+1}{4u^{1/3}}\,du$

$= \displaystyle\frac{1}{4}\int\left(u^{2/3} + u^{-1/3}\right)du$

when $x = 1$, $u = 2(1)-1 = 1$

when $x = 4.5$, $u = 2(4.5)-1 = 8$

$\displaystyle\int_1^{4.5} \frac{x}{\sqrt[3]{2x-1}}\,dx = \frac{1}{4}\int_1^8 \left(u^{2/3} + u^{-1/3}\right)du$

$= \dfrac{1}{4}\left[\dfrac{u^{5/3}}{5/3} + \dfrac{u^{2/3}}{2/3}\right]_1^8$

$= \dfrac{1}{4}\left[\dfrac{3u^{5/3}}{5} + 3\dfrac{u^{2/3}}{2}\right]_1^8$

$= \dfrac{3}{4}\left[\dfrac{u^{5/3}}{5} + \dfrac{u^{2/3}}{2}\right]_1^8$

$= \dfrac{3}{4}\left[\left(\dfrac{32}{5} + \dfrac{4}{2}\right) - \left(\dfrac{1}{5} + \dfrac{1}{2}\right)\right]$

$= 5.775$ units^3

is required area.