# MEI CORE 3 JUNE 2014

1)
$$\int_0^{\frac{\pi}{6}} (1-\sin 3x)\,dx = \left[x + \tfrac{1}{3}\cos 3x\right]_0^{\frac{\pi}{6}}$$
$$= \left(\frac{\pi}{6} + \frac{1}{3}\cos\frac{\pi}{2}\right) - \left(0 + \frac{1}{3}\cos 0\right)$$
$$= \frac{\pi}{6} - \frac{1}{3}$$

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2)
$$y = \ln(1-\cos 2x)$$
$$\Rightarrow \frac{dy}{dx} = \frac{2\sin 2x}{1-\cos 2x}$$

when $x = \frac{\pi}{6}$
$$\frac{dy}{dx} = \frac{2\sin\frac{\pi}{3}}{1-\cos\frac{\pi}{3}} = \frac{2 \times \frac{\sqrt{3}}{2}}{1-\frac{1}{2}} = \frac{\sqrt{3}}{\frac{1}{2}} = 2\sqrt{3}$$

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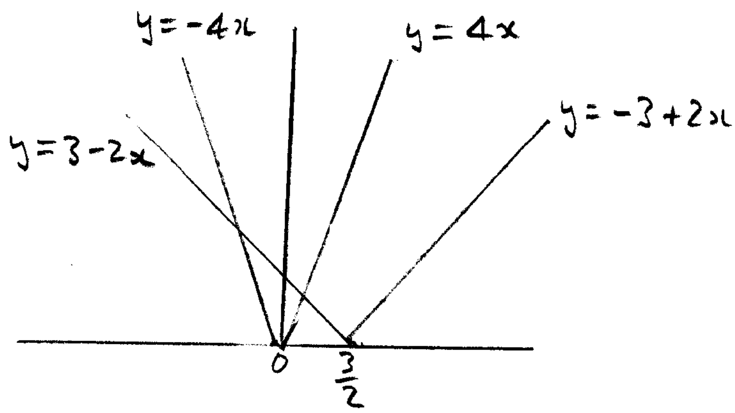
3)
$$|3-2x| = 4|x|$$




Solve for $x$ $\quad y = 3-2x$, $y = -4x$

$$\Rightarrow -4x = 3-2x$$
$$-3 = 2x$$
$$-\frac{3}{2} = x$$

Solve $\quad y = 3-2x$, $y = 4x$

$$\Rightarrow 4x = 3-2x$$
$$6x = 3$$
$$x = \frac{1}{2}$$

Solution $x = \frac{1}{2}$, $x = -\frac{3}{2}$

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4) $f(x) = a + \cos bx \qquad 0 \leq x \leq 2\pi$

i) $x = 0$

$$3 = a + \cos 0$$

$$3 = a + 1$$

$$a = 2$$

$x = 2\pi$

$$1 = 2 + \cos(2\pi b)$$

$$\Rightarrow \cos(2\pi b) = -1$$

$$\Rightarrow 2\pi b = \pi$$

$$\Rightarrow b = \tfrac{1}{2}$$

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ii) Let $y = 2 + \cos \frac{x}{2}$

swap variables

$$x = 2 + \cos \frac{y}{2}$$

make y subject

$$x - 2 = \cos \frac{y}{2}$$

$$\cos^{-1}(x-2) = \frac{y}{2}$$

$$2\cos^{-1}(x-2) = y$$

$$\therefore f^{-1}(x) = 2\cos^{-1}(x-2)$$

domain $1 \leq x \leq 3$

range $0 \leq f^{-1}(x) \leq 2\pi$

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5) $V = \frac{4}{3}\pi r^3 \qquad \frac{dV}{dt} = 10 \text{ cm}^3 s^{-1}$

Find $\frac{dr}{dt}$ when $r = 8$

$$\frac{dV}{dr} = 4\pi r^2$$

5 cont)

$$\frac{dr}{dt} = \frac{dr}{dV} \cdot \frac{dV}{dt} = \frac{\frac{dV}{dt}}{\frac{dV}{dr}} = \frac{10}{4\pi r^2}$$

when $r = 8$, $\frac{dr}{dt} = \frac{10}{4\pi \times 8^2}$ $cm\,s^{-1}$

$\frac{dr}{dt} = 0.0124$ $cm\,s^{-1}$ to 3 s.f.

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6) $V = Ae^{-kt}$

i) $V = 20000e^{-0.2t}$

$t = 0$ $V = £20000$

$t = 1$ $V = 20000e^{-0.2} = £16374.62$

Loss in value $= £3600$ to nearest £100

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ii) $t = 0$ $V = £15000$

$t = 1$ $V = £15000 - £2000 = £13000$

$13000 = 15000e^{-k \times 1}$

$\frac{13000}{15000} = e^{-k}$

$\ln\left(\frac{13}{15}\right) = -k$

$k = -\ln\left(\frac{13}{15}\right) = 0.14310 = 0.143$

to 3 s.f.

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6iii)

$$15000e^{-0.143t} = 20000e^{-0.2t}$$

$$\frac{e^{-0.143t}}{e^{-0.2t}} = \frac{20000}{15000} = \frac{4}{3}$$

$$e^{(-0.143t - -0.2t)} = \frac{4}{3}$$

$$e^{0.057t} = \frac{4}{3}$$

$$0.057t = \ln\left(\tfrac{4}{3}\right)$$

$$t = \frac{\ln\left(\tfrac{4}{3}\right)}{0.057}$$

$t = 5.047$ years approx 5 years

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7) i) 25 and 27 are consecutive odd numbers but neither is prime. So statement is false.

ii) If $m, n$ are consecutive even numbers we can write $m = 2k$, $n = 2k+2$ for some integer $k$

$$\therefore\ mn = 2k(2k+2)$$
$$= 4k(k+1)$$

Either $k$ or $k+1$ must be even so $4k(k+1)$ has a factor of $4 \times 2 = 8$ $\therefore$ $mn$ is divisible by 8

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8)

i)

$$y = f(x) = \frac{x}{\sqrt{2+x^2}}$$

$$f(-x) = \frac{-x}{\sqrt{2+(-x)^2}} = \frac{-x}{\sqrt{2+x^2}} = -\frac{x}{\sqrt{2+x^2}} = -f(x)$$

for all $x \in \mathbb{R}$

$\therefore$ $f(x)$ is an odd function

The function has rotational symmetry of order 2 about the origin $(0,0)$

ii)

$$f'(x) = \frac{\sqrt{2+x^2} \times 1 - x \times \frac{1}{2}(2+x^2)^{-\frac{1}{2}} \times 2x}{\sqrt{2+x^2}^{\,2}}$$

$$f'(x) = \frac{\sqrt{2+x^2} - \frac{x^2}{\sqrt{2+x^2}}}{(2+x^2)}$$

$$f'(x) = \frac{2+x^2 - x^2}{(2+x^2)^{3/2}}$$

$$f'(x) = \frac{2}{(2+x^2)^{3/2}}$$

At origin $f'(0) = \frac{2}{2^{3/2}} = \frac{1}{\sqrt{2}}$

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8iii)

$$\int_0^1 f(x)\,dx = \int_0^1 \frac{x}{\sqrt{2+x^2}}\,dx$$

Let $u = 2+x^2$

$$\frac{du}{dx} = 2x$$

$$du = 2x\,dx$$

$$\tfrac{1}{2}\,du = x\,dx$$

when $x = 1$ $u = 3$

when $x = 0$ $u = 2$

Integral becomes

$$\int_2^3 \frac{1}{\sqrt{u}}\frac{1}{2}\,du = \left[\frac{1}{2}\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]_2^3$$

$$= \left[u^{\frac{1}{2}}\right]_2^3$$

$$= \sqrt{3} - \sqrt{2} \text{ units}^2$$

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8 iv)

A)

$$y = \frac{x}{\sqrt{2+x^2}}$$

$$\Rightarrow y^2 = \frac{x^2}{2+x^2}$$

$$\Rightarrow \frac{1}{y^2} = \frac{2+x^2}{x^2}$$

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8 iv cont)

$$\Rightarrow \quad \frac{1}{y^2} = \frac{2}{x^2} + 1$$

B)

$$y^{-2} = 2x^{-2} + 1$$

$$-2y^{-3}\frac{dy}{dx} = -4x^{-3}$$

$$y^{-3}\frac{dy}{dx} = 2x^{-3}$$

$$\frac{1}{y^3}\frac{dy}{dx} = \frac{2}{x^3}$$

$$\frac{dy}{dx} = \frac{2y^3}{x^3}$$

Cannot be used to find gradient at origin because $\frac{0}{0}$ is undefined

9)

$$y = xe^{-2x} \qquad y = mx \qquad 0 < m < 1$$

i) At P

$$mx = xe^{-2x}$$

$$m = e^{-2x}$$

$$\ln m = -2x$$

$$\Rightarrow \quad x = -\frac{1}{2}\ln m$$

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9 ii) $$y = xe^{-2x}$$

$$\frac{dy}{dx} = x \times (-2e^{-2x}) + e^{-2x} \times 1$$

$$\frac{dy}{dx} = e^{-2x}(1-2x)$$

At P, $x = -\frac{1}{2}\ln m$, gradient of tgt $= \frac{dy}{dx}$

$$= e^{-2(-\frac{1}{2}\ln m)}\left(1-2\left(-\frac{1}{2}\ln m\right)\right)$$

$$= e^{\ln m}(1+\ln m)$$

$$= m(1+\ln m)$$

iii) Equally inclined so gradient of tgt = − gradient of OP

$$\Rightarrow \quad m(1+\ln m) = -m$$

$$\Rightarrow \quad 1+\ln m = -\frac{m}{m}$$

$$\Rightarrow \quad 1+\ln m = -1$$

$$\Rightarrow \quad \ln m = -2$$

$$\Rightarrow \quad m = e^{-2}$$

$P\left(-\frac{1}{2}\ln m, -\frac{1}{2}m\ln m\right)$ using $y = mx$

$P\left(-\frac{1}{2}\ln(e^{-2}), -\frac{1}{2}e^{-2}\ln(e^{-2})\right)$

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9iii cont) $P\left(1, \frac{1}{e^2}\right)$

iv $\text{Area} = \int_0^1 x e^{-2x}\,dx$ — area of $\Delta$ under OP

$= \int_0^1 x e^{-2x}\,dx - \frac{1}{2} \times 1 \times \frac{1}{e^2}$

$= \int_0^1 x e^{-2x}\,dx - \frac{1}{2e^2}$

$\int_0^1 x e^{-2x}\,dx$ Let $u = x$ Let $\frac{dv}{dx} = e^{-2x}$

$\Rightarrow \frac{du}{dx} = 1$ $\Rightarrow v = -\frac{1}{2}e^{-2x}$

$\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$

$$\int_0^1 x e^{-2x}\,dx = \left[-\frac{1}{2}xe^{-2x}\right]_0^1 + \int_0^1 \frac{1}{2}e^{-2x}\,dx$$

$$= \left[-\frac{1}{2}xe^{-2x}\right]_0^1 + \left[-\frac{1}{4}e^{-2x}\right]_0^1$$

$$= \left(-\frac{1}{2}e^{-2} - 0\right) + \left(-\frac{1}{4}e^{-2} - -\frac{1}{4}\right)$$

$$= -\frac{1}{2e^2} - \frac{1}{4e^2} + \frac{1}{4}$$

$$= -\frac{3}{4e^2} + \frac{1}{4}$$

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9iv cont) So area is given by

$$-\frac{3}{4e^2} + \frac{1}{4} - \frac{1}{2e^2}$$

$$= -\frac{5}{4e^2} + \frac{1}{4} \quad \text{units}^2$$