1) $$\frac{1+2x}{(1-2x)^2} = (1+2x)(1-2x)^{-2}$$

$$\approx (1+2x)\left[1 + -2(-2x) + \frac{-2.-3}{1.2}(-2x)^2\right]$$

$$= (1+2x)\left[1 + 4x + 12x^2\right]$$

$$\approx 1 + 4x + 12x^2 + 2x + 8x^2$$

$$= 1 + 6x + 20x^2$$

Valid for $|2x| < 1$

$$\Rightarrow -\tfrac{1}{2} < x < \tfrac{1}{2}$$

2) $$\cot 2\theta = \frac{1}{\tan 2\theta} = \frac{1}{\frac{2\tan\theta}{1-\tan^2\theta}}$$

$$\therefore \cot 2\theta = \frac{1-\tan^2\theta}{2\tan\theta}$$

$$\cot 2\theta = 1 + \tan\theta$$

$$\frac{1-\tan^2\theta}{2\tan\theta} = 1 + \tan\theta$$

$$1 - \tan^2\theta = (1+\tan\theta)2\tan\theta$$

$$1 - \tan^2\theta = 2\tan\theta + 2\tan^2\theta$$

$$0 = 3\tan^2\theta + 2\tan\theta - 1$$

$$0 = (3\tan\theta - 1)(\tan\theta + 1)$$

$$\Rightarrow \tan\theta = \frac{1}{3} \text{ or } \tan\theta = -1$$

When $\tan\theta = \frac{1}{3}$, $\theta = 18.4^\circ, 198.4^\circ$

When $\tan\theta = -1$, $\theta = 135^\circ, 315^\circ$

Solution for $0 \le \theta \le 360^\circ$

$$\theta = 18.4^\circ, 135^\circ, 198.4^\circ, 315^\circ$$

3) $$x = e^{2t} \qquad y = \frac{2t}{1+t}$$

i) $$\frac{dx}{dt} = 2e^{2t}$$

$$\frac{dy}{dt} = \frac{(1+t)2 - 2t(1)}{(1+t)^2}$$

$$\frac{dy}{dt} = \frac{2}{(1+t)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{2}{(1+t)^2}}{2e^{2t}}$$

$$\frac{dy}{dx} = \frac{1}{e^{2t}(1+t)^2}$$

When $t = 0$, $\frac{dy}{dx} = \frac{1}{1 \times 1^2}$

$$\frac{dy}{dx} = 1$$

3ii) $$x = e^{2t} \Rightarrow 2t = \ln x$$

$$t = \tfrac{1}{2}\ln x$$

$$\Rightarrow y = \frac{\ln x}{1 + \tfrac{1}{2}\ln x} = \frac{2\ln x}{2 + \ln x}$$

4) i) $A(1, 3, -2)$
$B(-1, 2, -3)$
$C(0, -8, 1)$

$\vec{AB} = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}$ $\vec{AC} = \begin{pmatrix} -1 \\ -11 \\ 3 \end{pmatrix}$

4 ii)

$\begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} . \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} = -4 + 1 + 3 = 0$
$\therefore \perp$

$\begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} . \begin{pmatrix} -1 \\ -11 \\ 3 \end{pmatrix} = -2 + 11 - 9 = 0$
$\therefore \perp$

$\begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$ is $\perp$ to two non-parallel lines in plane ABC

It is therefore $\perp$ to plane

Plane of form $2x - y - 3z = d$

A on plane so
$2(1) - (3) - 3(-2) = d$
$2 - 3 + 6 = d$
$5 = d$

Plane ABC is $2x - y - 3z = 5$

5) i) $\underline{r} = \begin{pmatrix} -5 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$

Verify $(1, 3, 2)$ on line

$-5 + 3\lambda = 1$
$\Rightarrow 3\lambda = 6$
$\lambda = 2$

When $\lambda = 2$

$\underline{r} = \begin{pmatrix} -5 + 3(2) \\ 3 + 0(2) \\ 4 - 1(2) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ ✓

$\underline{r} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

Verify $(1, 3, 2)$ on line

$-1 + 2\mu = 1$
$\Rightarrow 2\mu = 2$
$\Rightarrow \mu = 1$

When $\mu = 2$

$\underline{r} = \begin{pmatrix} -1 + 2(1) \\ 4 - 1(1) \\ 2 + 0(1) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ ✓

$\therefore$ lines meet at $(1, 3, 2)$

5ii) $\cos\theta = \dfrac{\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} . \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}}{\left|\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}\right| \left|\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}\right|}$

$\cos\theta = \dfrac{6 + 0 + 0}{\sqrt{10}\sqrt{5}} = \dfrac{6}{\sqrt{50}}$

$\theta = 31.9°$

6)i) $\angle BAC = 120 - (90-\theta) - 90$
$= \theta - 60^\circ$

$h = BC + CD = BC + AE$

$h = b\sin(\angle BAC) + a\sin\theta$

$h = b\sin(\theta - 60^\circ) + a\sin\theta$

6ii) $h = a\sin\theta + b(\sin\theta\cos 60^\circ - \cos\theta\sin 60^\circ)$

$h = a\sin\theta + \frac{b}{2}\sin\theta - \frac{\sqrt{3}}{2}b\cos\theta$

$h = \left(a + \frac{b}{2}\right)\sin\theta - \frac{\sqrt{3}}{2}b\cos\theta$

6iii) When OB is horizontal $h = 0$

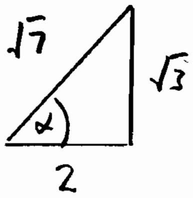
$\Rightarrow \left(a+\frac{b}{2}\right)\sin\theta - \frac{\sqrt{3}}{2}b\cos\theta = 0$

$\left(a+\frac{b}{2}\right)\sin\theta = \frac{\sqrt{3}b\cos\theta}{2}$

$\Rightarrow \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{3}b/2}{a + b/2}$

$\tan\theta = \frac{\sqrt{3}b}{2a+b}$

6iv) $2\sin\theta - \sqrt{3}\cos\theta$



$= \sqrt{7}\left(\frac{2}{\sqrt{7}}\sin\theta - \frac{\sqrt{3}}{\sqrt{7}}\cos\theta\right)$

$= \sqrt{7}\sin(\theta - \alpha)$

where $\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{2}\right) = 40.9^\circ$

$= \sqrt{7}\sin(\theta - 40.9^\circ)$

$h = \sqrt{7}\sin(\theta - 40.9^\circ)$

Max value of $h = \sqrt{7}$ m

Occurs when $\theta - 40.9^\circ = 90$

$\Rightarrow \theta = 130.9^\circ$

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7) $\frac{dx}{dt} = x(1-x)$

i) Consider $x = \frac{1}{1+e^{-t}}$

when $t = 0$, $x = \frac{1}{1+1} = 0.5$

$\therefore$ initial condition satisfied

If $x = \frac{1}{1+e^{-t}} = (1+e^{-t})^{-1}$

$\frac{dx}{dt} = -1(1+e^{-t})^{-2} \times (-e^{-t})$

$= \frac{e^{-t}}{(1+e^{-t})^2}$ (*)

Also

$x(1-x) = x - x^2$

$= \frac{1}{1+e^{-t}} - \frac{1}{(1+e^{-t})^2}$

$= \frac{1+e^{-t} - 1}{(1+e^{-t})^2} = \frac{e^{-t}}{(1+e^{-t})^2}$

This is the same as (*)

$\therefore$ $x = \frac{1}{1+e^{-t}}$ satisfies the diff eqn.

7ii) Require $x = \frac{3}{4}$

$$\frac{3}{4} = \frac{1}{1+e^{-t}}$$

$$3(1+e^{-t}) = 4$$

$$1 + e^{-t} = \frac{4}{3}$$

$$e^{-t} = \frac{1}{3}$$

$$-t = \ln\frac{1}{3}$$

$$t = \ln 3$$

$t = 1.10$ years

7iii)

$$\frac{1}{x^2(1-x)} \equiv \frac{A}{x^2} + \frac{B}{x} + \frac{C}{1-x}$$

$$1 = A(1-x) + Bx(1-x) + Cx^2$$

$x = 1$

$\Rightarrow 1 = C$

$x = 0$

$\Rightarrow 1 = A$

Coeff of x^2

$0 = -B + C$

$\Rightarrow B = C \quad \therefore \quad B = 1$

$$\frac{1}{x^2(1-x)} \equiv \frac{1}{x^2} + \frac{1}{x} + \frac{1}{1-x}$$

7iv)

$$\frac{dx}{dt} = x^2(1-x)$$

$$\Rightarrow \int \frac{1}{x^2(1-x)}\,dx = \int 1\,dt$$

$$\int\left(\frac{1}{x^2} + \frac{1}{x} + \frac{1}{1-x}\right)dx = \int 1\,dt$$

$$-\frac{1}{x} + \ln x - \ln(1-x) + c = t$$

$t = 0$, $x = 0.5$ gives

$$-2 + \ln\tfrac{1}{2} - \ln\tfrac{1}{2} + c = 0$$

$\Rightarrow c = 2$

$$\therefore \quad t = -\frac{1}{x} + \ln\left(\frac{x}{1-x}\right) + 2$$

7v) Require $x = \frac{3}{4}$

$$t = -\frac{4}{3} + \ln\left(\frac{\frac{3}{4}}{1-\frac{3}{4}}\right) + 2$$

$$t = \frac{2}{3} + \ln(3)$$

$t = 1.77$ years