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1) $\int_{-2}^{2} \sqrt{1+e^x}\,dx$

x	-2	-1	0	1	2
y	1.0655	1.1696	1.4142	1.9283	2.8964

i)

Integral = Area under curve $\approx \frac{h}{2}\left[y_0 + 2(y_1+y_2+y_3) + y_4\right]$

$= \frac{1}{2}\left[1.0655 + 2(1.1696+1.4142+1.9283) + 2.8964\right]$

$= 6.49305$

$$\int_{-2}^{2} \sqrt{1+e^x} \approx 6.49$$

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ii) Trapezia all partly above curve so over-estimate area

More strips would mean less trapezia area above curve

so would give more accurate smaller estimate.

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2) $x = \dfrac{1}{1+t}$ $\qquad y = \dfrac{1-t}{1+2t}$

$(1+t)x = 1$

$x + tx = 1$

$tx = 1-x$

$t = \dfrac{1-x}{x}$

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2 cont) Sub for t in $y = \dfrac{1-t}{1+2t}$

$$y = \frac{1 - \frac{1-x}{x}}{1 + \frac{2(1-x)}{x}} \qquad \times \frac{x}{x}$$

$$y = \frac{x - (1-x)}{x + 2(1-x)}$$

$$y = \frac{2x-1}{2-x}$$

3) $$\frac{1}{(3-2x)^3} = \frac{1}{\left(3\left(1-\frac{2x}{3}\right)\right)^3} = \frac{1}{27}\left(1-\frac{2x}{3}\right)^{-3}$$

$$\approx \frac{1}{27}\left[1 + -3\left(-\frac{2x}{3}\right) + \frac{-3.-4}{1.2}\left(-\frac{2x}{3}\right)^2 \ldots\right]$$

$$\approx \frac{1}{27}\left[1 + 2x + \frac{8}{3}x^2\right]$$

$$\approx \frac{1}{27} + \frac{2}{27}x + \frac{8}{81}x^2$$

Valid for $\left|\frac{2x}{3}\right| < 1$

$$|2x| < 3$$

$$|x| < \frac{3}{2}$$

$$-\frac{3}{2} < x < \frac{3}{2}$$

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4) $A(2,0,-1)$

$B(4,3,-6)$

i) $C(9,3,-4)$

$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$ $\overrightarrow{BC} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}$

$$\overrightarrow{AB}.\overrightarrow{BC} = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}.\begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} = 10 + 0 - 10 = 0$$

$\therefore$ $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are $\perp$

ii)
$$\text{Area} = \frac{1}{2}|BC| \times |AB|$$
$$= \frac{1}{2} \times \sqrt{2^2+3^2+(-5)^2} \times \sqrt{5^2+0^2+2^2}$$
$$= \frac{1}{2} \times \sqrt{38} \times \sqrt{29}$$
$$= 16.6 \text{ units}^2 \quad \text{to 3 s.f.}$$

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5) Show $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$

$$\frac{\sin 2\theta}{1+\cos 2\theta} = \frac{2\sin\theta\cos\theta}{1+2\cos^2\theta-1} = \frac{2\sin\theta\cos\theta}{2\cos^2\theta}$$
$$= \frac{\sin\theta}{\cos\theta} = \tan\theta$$

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6 i) $\underline{r} = \begin{pmatrix} -8 \\ -2 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$ Plane $2x - 3y + z = 11$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -8 - 3\lambda \\ -2 \\ 6 + \lambda \end{pmatrix}$$

Sub in plane

$$2(-8-3\lambda) - 3(-2) + (6+\lambda) = 11$$

$$-16 - 6\lambda + 6 + 6 + \lambda = 11$$

$$-5\lambda - 4 = 11$$

$$-5\lambda = 15$$

$$\lambda = -3$$

Sub in line

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -8 - 3(-3) \\ -2 \\ 6 + (-3) \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

Point of intersection is $(1, -2, 3)$

ii) normal $= \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

$$\cos\theta = \frac{\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}}{\left|\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}\right| \left|\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}\right|}$$

$$\cos\theta = \frac{-6 + 0 + 1}{\sqrt{14}\sqrt{10}} = \frac{-5}{\sqrt{140}}$$

$\theta = 115.0^\circ$ Acute angle $= 180 - 115.0 = 65.0^\circ$

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7 i) $v = 5(1 - e^{-2t})$

When $t = 0$, $v = 5(1 - e^{0}) = 5(1-1) = 0$

As $t \to \infty$, $v \to 5(1 - e^{-\infty}) = 5(1-0) = 5$

When $t = 0.5$

$v = 5(1 - e^{-1}) = 3.16 \text{ ms}^{-1}$ (3 s.f)

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ii) $v = 5(1 - e^{-2t})$

$\frac{dv}{dt} = 10e^{-2t} = 10 - 10 + 10e^{-2t}$

$= 10 - 10(1 - e^{-2t})$

$= 10 - 2v$

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iii) $\frac{dv}{dt} = 10 - 0.4v^2$

$\frac{dv}{dt} = \frac{1}{10}(100 - 4v^2)$

$\frac{dv}{dt} = \frac{1}{10}(10 + 2v)(10 - 2v)$

$\frac{dv}{dt} = \frac{4}{10}(5 + v)(5 - v)$

$10\frac{dv}{dt} = 4(5 + v)(5 - v)$

$\frac{10}{(5+v)(5-v)}\frac{dv}{dt} = 4$

$\frac{10}{(5+v)(5-v)} \equiv \frac{A}{5+v} + \frac{B}{5-v}$

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7iii cont)

$$10 \equiv A(5-v) + B(5+v)$$

$v=5$

$$10 = 10B \Rightarrow B = 1$$

$v=-5$

$$10 = 10A \Rightarrow A = 1$$

$$\frac{10}{(5+v)(5-v)} \equiv \frac{1}{5+v} + \frac{1}{5-v}$$

$$\therefore \left(\frac{1}{5+v} + \frac{1}{5-v}\right)\frac{dv}{dt} = 4$$

$$\int\left(\frac{1}{5+v} + \frac{1}{5-v}\right)dv = \int 4\,dt$$

$$\ln(5+v) - \ln(5-v) = 4t + c$$

$v=0, t=0$

$$\ln 5 - \ln 5 = 0 + c \qquad \Rightarrow c = 0$$

$$\therefore \ln\left(\frac{5+v}{5-v}\right) = 4t$$

$$t = \frac{1}{4}\ln\left(\frac{5+v}{5-v}\right)$$

iv) $v = \dfrac{5(1-e^{-4t})}{1+e^{-4t}}$

As $t \to \infty$ $v \to \dfrac{5(1-0)}{1+0} = 5$

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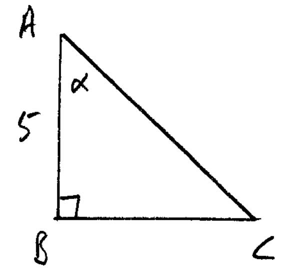
7iv) cont) when $t = 0.5$ $\quad v = \dfrac{5(1-e^{-2})}{1+e^{-2}} = 3.81\,\text{ms}^{-1}$ to 3s.f.

7v) The first model as 3.16 is closer to 3 than 3.81 is.

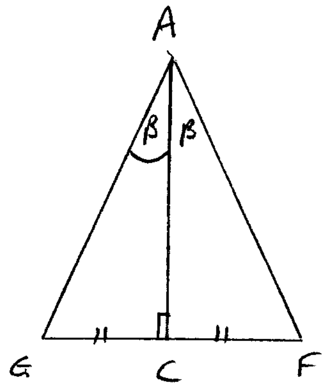
8) i)



$\cos\alpha = \dfrac{AB}{AC} = \dfrac{5}{AC}$

$\Rightarrow AC\cos\alpha = 5$

$\Rightarrow AC = \dfrac{5}{\cos\alpha}$ or $5\sec\alpha$



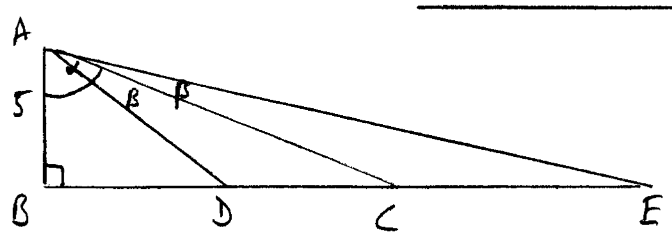
$GF = 2GC$

$\tan\beta = \dfrac{GC}{AC}$

$\Rightarrow GC = AC\tan\beta = 5\sec\alpha\tan\beta$

$\Rightarrow GF = 10\sec\alpha\tan\beta$

ii)



$CE = BE - BC$

$\tan(\alpha+\beta) = \dfrac{BE}{5} \quad \Rightarrow \quad BE = 5\tan(\alpha+\beta)$

$\tan\alpha = \dfrac{BC}{5} \quad \Rightarrow \quad BC = 5\tan\alpha$

$\therefore CE = 5\tan(\alpha+\beta) - 5\tan\alpha = 5\left(\tan(\alpha+\beta) - \tan\alpha\right)$

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8 ii) cont) $\Rightarrow CE = 5\left[\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} - \tan\alpha\right]$

$$= 5\left[\frac{\tan\alpha + \tan\beta - \tan\alpha(1 - \tan\alpha\tan\beta)}{1 - \tan\alpha\tan\beta}\right]$$

$$= 5\left[\frac{\tan\beta + \tan^2\alpha\tan\beta}{1 - \tan\alpha\tan\beta}\right]$$

$$= 5\left[\frac{\tan\beta(1 + \tan^2\alpha)}{1 - \tan\alpha\tan\beta}\right]$$

$$= \frac{5\tan\beta\sec^2\alpha}{1 - \tan\alpha\tan\beta}$$

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iii) $\alpha = 45° \Rightarrow \tan\alpha = 1, \quad \sec^2\alpha = 1 + \tan^2\alpha = 2$

$$CE = \frac{5t(2)}{1-t} = \frac{10t}{1-t}$$

$$CD = \frac{10t}{1+t}$$

Now $DE = CD + CE$

$$= \frac{10t}{1+t} + \frac{10t}{1-t} = \frac{10t(1-t) + 10t(1+t)}{(1+t)(1-t)}$$

$$= \frac{10t - 10t^2 + 10t + 10t^2}{1-t^2} = \frac{20t}{1-t^2}$$

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8iv) $GF = 10 \sec\alpha \tan\beta$ (from part i)

$= 10\sqrt{1+\tan^2\alpha}\ \tan\beta$

$= 10\sqrt{2}\ t$

8v) Given $DE = 2GF$

$\Rightarrow \dfrac{20t}{1-t^2} = 20\sqrt{2}\ t$

($\div 20t$)

$\Rightarrow \dfrac{1}{1-t^2} = \sqrt{2}$

$\Rightarrow 1 = \sqrt{2}(1-t^2)$

$1 = \sqrt{2} - \sqrt{2}t^2$

$\sqrt{2}t^2 = \sqrt{2} - 1$

$t^2 = \dfrac{\sqrt{2}-1}{\sqrt{2}} = 1 - \dfrac{1}{\sqrt{2}}$

$\tan\beta = t$

so $\beta = \tan^{-1} t = \tan^{-1}\left(\sqrt{1-\frac{1}{\sqrt{2}}}\right)$

$\beta = 28.4^\circ$