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1) $$\frac{2x}{x+1} - \frac{1}{x-1} = 1$$

$$2x(x-1) - 1(x+1) = 1(x+1)(x-1)$$

$$2x^2 - 2x - x - 1 = x^2 - 1$$

$$2x^2 - 2x - x - 1 - x^2 + 1 = 0$$

$$x^2 - 3x = 0$$

$$x(x-3) = 0$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad x = 3$$

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2) $$\sqrt[3]{1-2x} = (1-2x)^{\frac{1}{3}}$$

Expansion of $(1+x)^n = 1 + nx + \frac{n.(n-1)}{1.2}x^2 + \ldots$

$$(1-2x)^{\frac{1}{3}} = 1 + \tfrac{1}{3}(-2x) + \frac{\frac{1}{3}.-\frac{2}{3}}{1.2}(-2x)^2 + \frac{\frac{1}{3}.-\frac{2}{3}.-\frac{5}{3}}{1.2.3}(-2x)^3 + +$$

$$= 1 - \frac{2}{3}x - \frac{4}{9}x^2 - \frac{40}{81}x^3 - \ldots\ldots$$

Valid for $|2x| < 1$

$$\Rightarrow \quad |x| < \tfrac{1}{2}$$

$$\Rightarrow -\tfrac{1}{2} < x < \tfrac{1}{2}$$

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3) $x = \sin\theta$, $y = \sin 2\theta$ $\qquad 0 \le \theta \le 2\pi$

i) $\dfrac{dy}{dx} = \dfrac{dy}{d\theta} \Big/ \dfrac{dx}{d\theta}$ $\qquad \dfrac{dy}{d\theta} = 2\cos 2\theta$

$\dfrac{dx}{d\theta} = \cos\theta$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{2\cos 2\theta}{\cos\theta}$

when $\theta = \frac{\pi}{6}$, $\dfrac{dy}{dx} = \dfrac{2\cos\frac{\pi}{3}}{\cos\frac{\pi}{6}} = \dfrac{2 \times \frac{1}{2}}{\frac{\sqrt{3}}{2}}$

$\dfrac{dy}{dx} = \dfrac{2}{\sqrt{3}}$ or $\dfrac{2\sqrt{3}}{3}$

3 ii) $y = \sin 2\theta$, $x = \sin\theta$

$y = 2\sin\theta\cos\theta$

$y = 2\sin\theta\sqrt{1-\sin^2\theta}$

$y = 2x\sqrt{1-x^2}$

$y^2 = 4x^2(1-x^2)$

$y^2 = 4x^2 - 4x^4$

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4a) $y = \sqrt{1+e^{2x}}$

$$\text{Volume} = \int_0^2 \pi y^2 \, dx$$

$$= \pi \int_0^2 (1+e^{2x}) \, dx$$

$$= \pi \left[x + \tfrac{1}{2} e^{2x} \right]_0^2$$

$$= \pi \left[(2 + \tfrac{1}{2}e^4) - (0 + \tfrac{1}{2}e^0) \right]$$

$$= \pi \left[2 + \tfrac{1}{2}e^4 - \tfrac{1}{2} \right]$$

$$= \pi \left(\tfrac{3}{2} + \tfrac{1}{2}e^4 \right)$$

$$= \frac{\pi}{2}(3 + e^4)$$

4b)

i)

x	0	0.5	1	1.5	2
y	1.4142	1.9283	2.8964	4.5919	7.4564

$$A \approx \frac{h}{2}\left(y_0 + 2(y_1 + y_2 + y_3) + y_4\right)$$

$$A \approx \frac{0.5}{2}\left(1.4142 + 2(1.9283 + 2.8964 + 4.5919) + 7.4564\right)$$

$$\text{Area} \approx 6.926$$

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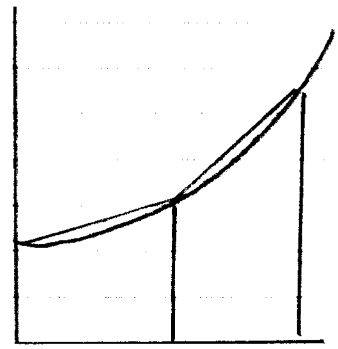
4bii) Curve



Trapezium rule gives overestimate of area for this curve. More strips reduces excess area above curve and so reduces overestimate

So 8 strips gives 6.823

16 strips gives 6.797

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5) $2\sec^2\theta = 5\tan\theta \qquad 0 \leq \theta \leq \pi$

$2(1+\tan^2\theta) = 5\tan\theta$

$2\tan^2\theta - 5\tan\theta + 2 = 0$

$(2\tan\theta - 1)(\tan\theta - 2) = 0$

$\Rightarrow \quad 2\tan\theta - 1 = 0 \qquad$ or $\qquad \tan\theta - 2 = 0$

$2\tan\theta = 1 \qquad\qquad \tan\theta = 2$

$\tan\theta = \frac{1}{2}$

$\Rightarrow \quad \theta = 26.6^\circ \qquad$ or $\qquad \theta = 63.4^\circ$

or $\theta = 0.464$ radians $\qquad$ or $\qquad \theta = 1.107$ radians

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6) i)

$\sin\theta = \dfrac{1}{AC} \Rightarrow AC\sin\theta = 1$

$\Rightarrow AC = \dfrac{1}{\sin\theta}$

$\cos\phi = \dfrac{AC}{AD} \Rightarrow AD\cos\phi = AC$

$\Rightarrow AD = \dfrac{AC}{\cos\phi}$

$\Rightarrow AD = \dfrac{1}{\sin\theta\cos\phi}$

ii)

$\sin(\theta+\phi) = \dfrac{DE}{AD}$

$\Rightarrow DE = AD \times \sin(\theta+\phi)$

$DE = \dfrac{1}{\sin\theta\cos\phi}\sin(\theta+\phi)$

$DE = \dfrac{\sin\theta\cos\phi + \cos\theta\sin\phi}{\sin\theta\cos\phi}$

$DE = \dfrac{\sin\theta\cos\phi}{\sin\theta\cos\phi} + \dfrac{\cos\theta\sin\phi}{\sin\theta\cos\phi}$

$DE = 1 + \cot\theta\tan\phi$

$DE = 1 + \dfrac{\tan\phi}{\tan\theta}$

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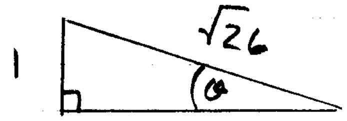
7 i)

$D(6, 0, 2)$
$E(1, 0, 3)$

$$|DE| = \sqrt{(6-1)^2 + (0-0)^2 + (2-3)^2} = \sqrt{26} \text{ m}$$
$$= 5.10 \text{ m}$$
to 3 s.f



Angle with horiz. $\theta = \sin^{-1} \frac{1}{\sqrt{26}} = 11.3^\circ$

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ii) $A(0, -4, 0)$
$D(6, 0, 2)$
$E(1, 0, 3)$

$$\vec{AD} = \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix} \qquad \vec{DE} = \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix} . \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix} = 6 - 16 + 10 = 0$$

$$\begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix} . \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} = -5 + 0 + 5 = 0$$

$\therefore \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix}$ is $\perp$ to two non-parallel vectors in plane ADE

It is $\therefore$ normal to plane

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7ii cont) Plane is of form

$$1x - 4y + 5z = d$$

$(0,-4,0)$ on plane so

$$-4(-4) = d$$

$$16 = d$$

Plane ADE is

$$x - 4y + 5z = 16$$

$B(8,-a,0)$ on plane

$$8 + 4a + 0 = 16$$

$$4a = 16 - 8$$

$$4a = 8$$

$$a = 2$$

7iii) $B(8,-2,0)$
$C(8,2,0)$
$D(6,0,2)$

Verify Plane BCD is $x + z = 8$

Verify B, C, D are on this plane

B $\quad 8 + 0 = 8$ ✓

C $\quad 8 + 0 = 8$ ✓

D $\quad 6 + 2 = 8$ ✓

$\therefore$ plane BCD is $x + z = 8$

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7iii cont) Angles between planes same as angles between normals

normal to plane ABDE $= \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix}$

normal to plane BCD $= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$$\cos\theta = \frac{\begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}{\left|\begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix}\right| \left|\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right|}$$

$$\cos\theta = \frac{1 + 0 + 5}{\sqrt{42}\sqrt{2}} = \frac{6}{\sqrt{84}}$$

$$\theta = \cos^{-1}\left(\frac{6}{\sqrt{84}}\right) = 49.1^\circ$$

Acute angle between planes $= 49.1^\circ$

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8 i) $$10\frac{dh}{dt} = 20 - h \qquad h=0,\ t=0$$

i) $\frac{dh}{dt} = 0$ when $h = 20$

Tree grows towards maximum height of 20m

ii) $$h = 20(1 - e^{-0.1t})$$

when $t = 0$, $h = 20(1-e^{0}) = 20(1-1) = 0$

so initial condition satisfied

$$\frac{dh}{dt} = 20(-0.1)(-e^{-0.1t}) = 2e^{-0.1t}$$

$$10\frac{dh}{dt} = 20e^{-0.1t}$$

but since $h = 20(1-e^{-0.1t})$

$$h = 20 - 20e^{-0.1t}$$

$$20e^{-0.1t} = 20 - h$$

$$\therefore\ 10\frac{dh}{dt} = 20 - h$$

Thus $h = 20(1-e^{-0.1t})$ satisfies diff eqn.

iii) $$200\frac{dh}{dt} = 400 - h^2$$

$$\Rightarrow \int \frac{200}{400-h^2}\,dh = \int 1\,dt$$

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8iii) cont)

$$\frac{200}{400-h^2} = \frac{200}{(20+h)(20-h)}$$

$$\frac{200}{(20+h)(20-h)} \equiv \frac{A}{20+h} + \frac{B}{20-h}$$

$$200 = A(20-h) + B(20+h)$$

$h=20$

$$200 = 40B \Rightarrow B = 5$$

$h=-20$

$$200 = 40A \Rightarrow A = 5$$

so $$\frac{200}{400-h^2} \equiv \frac{5}{20+h} + \frac{5}{20-h}$$

$$\Rightarrow \int \left(\frac{5}{20+h} + \frac{5}{20-h} \right) dh = \int 1 \, dt$$

$$\Rightarrow 5\ln(20+h) - 5\ln(20-h) = t + c$$

$$5\ln\left(\frac{20+h}{20-h}\right) = t + c$$

$h=0, t=0$ $$5\ln(1) = 0 + c$$

$$\Rightarrow 0 = c$$

$$\therefore 5\ln\left(\frac{20+h}{20-h}\right) = t$$

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8 iii) cont

$$\Rightarrow \ln\left(\frac{20+h}{20-h}\right) = 0.2t$$

$$\Rightarrow \frac{20+h}{20-h} = e^{0.2t}$$

$$\Rightarrow 20+h = (20-h)e^{0.2t}$$

$$20+h = 20e^{0.2t} - he^{0.2t}$$

$$h + he^{0.2t} = 20e^{0.2t} - 20$$

$$h(1+e^{0.2t}) = 20(e^{0.2t}-1)$$

$$h = \frac{20(e^{0.2t}-1)}{e^{0.2t}+1}$$

divide top and bottom by $e^{0.2t}$

$$h = \frac{20(1-e^{-0.2t})}{1+e^{-0.2t}}$$

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8 iv) As $t \to \infty$ $h \to \frac{20(1-0)}{1+0} = 20$

long term height $\to$ 20 m

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8 v) $h = 20(1-e^{-0.1}) = 1.90$ m

$$h = \frac{20(1-e^{-0.2})}{1+e^{-0.2}} = 1.99 \text{ m}$$

2nd model fits data better
