1) $\dfrac{x}{x^2-4} + \dfrac{2}{x+2}$

$= \dfrac{x + 2(x-2)}{(x+2)(x-2)}$

$= \dfrac{3x-4}{(x+2)(x-2)}$ or $\dfrac{3x-4}{x^2-4}$

2) $y = \sqrt{1+e^{2x}}$

Volume $= \int_0^1 \pi y^2 \, dx$

$= \pi \int_0^1 (1+e^{2x}) \, dx$

$= \pi \left[x + \frac{1}{2} e^{2x} \right]_0^1$

$= \pi \left[\left(1 + \frac{1}{2}e^2\right) - \left(0 + \frac{1}{2}\right) \right]$

$= \frac{\pi}{2}(1+e^2)$

3) $\cos 2\theta = \sin\theta \quad 0 \le \theta \le 2\pi$

$\Rightarrow 1 - 2\sin^2\theta = \sin\theta$

$\Rightarrow 0 = 2\sin^2\theta + \sin\theta - 1$

$\Rightarrow 0 = (2\sin\theta - 1)(\sin\theta + 1)$

$\Rightarrow \sin\theta = \frac{1}{2}$ or $\sin\theta = -1$

$\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

4) $x = 2\sec\theta, \quad y = 3\tan\theta$

$\Rightarrow \frac{x}{2} = \sec\theta, \quad \frac{y}{3} = \tan\theta$

Now $1 + \tan^2\theta = \sec^2\theta$

$\therefore \quad 1 + \frac{y^2}{3^2} = \frac{x^2}{2^2}$

$\Rightarrow \quad 1 = \frac{x^2}{4} - \frac{y^2}{9}$

5)

i) $x = 1 + u^2, \quad y = 2u^3$

$\frac{dx}{du} = 2u, \quad \frac{dy}{du} = 6u^2$

$\frac{dy}{dx} = \frac{dy}{du} \Big/ \frac{dx}{du} = \frac{6u^2}{2u} = 3u$

$\frac{dy}{dx} = 3u$

ii) At $(5, 16)$ $\quad u = 2$

$\therefore$ gradient $= 6$

6)

i) $\frac{1}{\sqrt{1+4x^2}} = (1+4x^2)^{-\frac{1}{2}}$

$\approx 1 + -\frac{1}{2}(4x^2) + \frac{-\frac{1}{2} \cdot -\frac{3}{2}}{1 \cdot 2}(4x^2)^2$

$= 1 - 2x^2 + 6x^4$

Valid for $|4x^2| < 1$

$-\frac{1}{2} < x < \frac{1}{2}$

6ii) $(1-x^2)(1+4x^2)^{-\frac{1}{2}}$

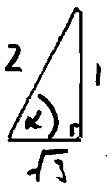
$\approx (1-x^2)(1-2x^2+6x^4)$

$\approx 1-2x^2+6x^4-x^2+2x^4$

$= 1-3x^2+8x^4\ldots$

7)

$\sqrt{3}\sin x - \cos x$



$=2\left(\frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x\right)$

$= 2\sin(x-\alpha)$

where $\alpha = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

Answer $= 2\sin\left(x-\frac{\pi}{6}\right)$

Max point when $x-\frac{\pi}{6} = \frac{\pi}{2}$

$\Rightarrow x = \frac{\pi}{2}+\frac{\pi}{6} = \frac{2\pi}{3}$

Max at $\left(\frac{2\pi}{3}, 2\right)$

8)

i) Subst A, B, C in eqn of plane to verify it is plane ABC

$3x + 2y + 20z + 300 = 0$

$A(0,0,-15)$

$0+0+20(-15)+300=0$ ✓

$B(100,0,-30)$

$300+0+20(-30)+300=0$

$300-600+300=0$ ✓

$C(0,100,-25)$

$0+2(100)+20(-25)+300=0$

$200-500+300=0$ ✓

∴ Plane is the plane ABC

ii)

$D(0,0,-40)$
$E(100,0,-50)$
$F(0,100,-35)$

$\overrightarrow{DE} = \begin{pmatrix}100\\0\\-10\end{pmatrix}$ $\overrightarrow{DF} = \begin{pmatrix}0\\100\\5\end{pmatrix}$

$\begin{pmatrix}100\\0\\-10\end{pmatrix}.\begin{pmatrix}2\\-1\\20\end{pmatrix} = 200-200=0$

$\begin{pmatrix}0\\100\\5\end{pmatrix}.\begin{pmatrix}2\\-1\\20\end{pmatrix} = -100+100=0$

∴ $\begin{pmatrix}2\\-1\\20\end{pmatrix}$ is $\perp$ to both $\overrightarrow{DE}$ and $\overrightarrow{DF}$

It is ∴ normal to plane DEF

Plane is of form

$2x - y + 20z = d$

$D(0,0,-40)$ on plane so

$0-0+20(-40) = d$

$-800 = d$

Plane DEF is

$2x - y + 20z = -800$

8iii) Angle between planes
= angle between their normals

$$\cos\theta = \frac{\begin{pmatrix}3\\2\\20\end{pmatrix}.\begin{pmatrix}2\\-1\\20\end{pmatrix}}{\left|\begin{pmatrix}3\\2\\20\end{pmatrix}\right|\left|\begin{pmatrix}2\\-1\\20\end{pmatrix}\right|}$$

$$\cos\theta = \frac{6-2+400}{\sqrt{9+4+400}\sqrt{4+1+400}}$$

$$\cos\theta = \frac{404}{\sqrt{413}\sqrt{405}}$$

$$\theta = 8.95^\circ$$

8iv)

$$\underline{r} = \begin{pmatrix}15\\34\\0\end{pmatrix} + \lambda\begin{pmatrix}3\\2\\20\end{pmatrix}$$

is eqn of line RS

At S

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}15+3\lambda\\34+2\lambda\\0+20\lambda\end{pmatrix}$$

S on plane ABC so

$$3(15+3\lambda)+2(34+2\lambda)+20(20\lambda)+300=0$$

$$45+9\lambda+68+4\lambda+400\lambda+300=0$$

$$413\lambda = -413$$

$$\lambda = -1$$

$$\underline{r}_S = \begin{pmatrix}15-3\\34-2\\-20\end{pmatrix} = \begin{pmatrix}12\\32\\-20\end{pmatrix}$$

S is point $(12, 32, -20)$

9)
i) $$\frac{dv}{dt} = 10e^{-\frac{1}{2}t}$$

$$v = \int 10e^{-\frac{1}{2}t}\,dt$$

$$v = -20e^{-\frac{1}{2}t} + c$$

$t=0,\ v=0$

$$0 = -20+c$$
$$20 = c$$

$$\therefore\ v = -20e^{-\frac{1}{2}t} + 20 \ \text{ms}^{-1}$$

ii) $v \to 20\ \text{ms}^{-1}$ as $t \to \infty$

iii) Let $\dfrac{1}{(w-4)(w+5)} \equiv \dfrac{A}{w-4} + \dfrac{B}{w+5}$

$\Rightarrow\ 1 \equiv A(w+5) + B(w-4)$

when $w=-5$ $\quad 1 = B(-5-4)$

$\Rightarrow B = -\frac{1}{9}$

when $w=4$ $\quad 1 = A(4+5)$

$\Rightarrow A = \frac{1}{9}$

$$\frac{1}{(w-4)(w+5)} \equiv \frac{1}{9(w-4)} - \frac{1}{9(w+5)}$$

iv)

$$\frac{dw}{dt} = -\frac{1}{2}(w-4)(w+5)$$

$$\Rightarrow \int \frac{1}{(w-4)(w+5)}\,dw = \int -\frac{1}{2}\,dt$$

$$\Rightarrow \int \left(\frac{1}{9(w-4)} - \frac{1}{9(w-5)}\right) dw = \int -\frac{1}{2}\,dt$$

$$\Rightarrow \frac{1}{9}\ln(w-4) - \frac{1}{9}\ln(w+5) = -\frac{1}{2}t + c$$

$$\Rightarrow \frac{1}{9}\ln\left(\frac{w-4}{w+5}\right) = -\frac{1}{2}t + c$$

$$\Rightarrow \ln\left(\frac{w-4}{w+5}\right) = -\frac{9t}{2} + c$$

$$\Rightarrow \frac{w-4}{w+5} = e^{-\frac{9t}{2}+c}$$

$$\Rightarrow \frac{w-4}{w+5} = Ae^{-\frac{9t}{2}}$$

Given $w = 10$ when $t = 0$

$$\therefore \frac{10-4}{10+5} = Ae^{0}$$

$$\Rightarrow \frac{6}{15} = A$$

$$\Rightarrow 0.4 = A$$

$$\therefore \frac{w-4}{w+5} = 0.4e^{-4.5t}$$

v)

As $t \to \infty$

$$\frac{w-4}{w+5} \to 0$$

$$\Rightarrow w - 4 \to 0$$

$$\Rightarrow w \to 4 \text{ ms}^{-1}$$