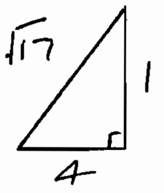
MEI CORE 4 JUNE 2009 ①

1) $4\cos\theta - \sin\theta$



$= \sqrt{17}\left(\frac{4}{\sqrt{17}}\cos\theta - \frac{1}{\sqrt{17}}\sin\theta\right)$

$= \sqrt{17}\cos(\theta+\alpha)$

where $\alpha = \tan^{-1}\frac{1}{4}$

$\alpha = 0.245$ radians

$\sqrt{17}\cos(\theta+0.245)$

$4\cos\theta - \sin\theta = 3$

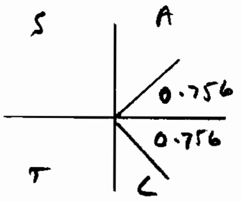
$\sqrt{17}\cos(\theta+0.245) = 3$

$\cos(\theta+0.245) = \frac{3}{\sqrt{17}}$

$\theta + 0.245 = \cos^{-1}\left(\frac{3}{\sqrt{17}}\right)$

$\theta + 0.245 = 0.756,\ 5.527$

$\theta = 0.511$ radians, 5.282 radians



0.756

$2\pi - 0.756$

2) $\frac{x}{(x+1)(2x+1)} \equiv \frac{A}{x+1} + \frac{B}{2x+1}$

$x \equiv A(2x+1) + B(x+1)$

$x=-1$

$\Rightarrow -1 = A(-1) \Rightarrow A = 1$

$x = -\frac{1}{2}$

$\Rightarrow -\frac{1}{2} = B\left(+\frac{1}{2}\right) \Rightarrow B = -1$

$\int \frac{x}{(x+1)(2x+1)}\,dx = \int\left(\frac{1}{x+1} - \frac{1}{2x+1}\right)dx$

$= \ln(x+1) - \frac{1}{2}\ln(2x+1) + c$

$= \ln(x+1) - \ln(2x+1)^{\frac{1}{2}} + c$

$= \ln\left(\frac{x+1}{\sqrt{2x+1}}\right) + c$

3) $\frac{dy}{dx} = 3x^2y$

$\Rightarrow \int \frac{1}{y}\,dy = \int 3x^2\,dx$

$\ln y = x^3 + c$

thro $(1,1)$

$\Rightarrow \ln 1 = 1^3 + c$

$0 = 1 + c$

$-1 = c$

$\therefore \ln y = x^3 - 1$

$y = e^{(x^3-1)}$

4) $V = \pi\int_0^4 x^2\,dy$

(since $y = 4$ when $x = 0$)

$y = 4 - x^2 \Rightarrow x^2 = 4 - y$

$\therefore V = \pi\int_0^4 (4-y)\,dy$

$V = \pi\left[4y - \frac{y^2}{2}\right]_0^4$

4 cont) $V = \pi\left[(16 - 8) - (0 - 0)\right]$

$V = 8\pi$

5) $x = at^3 \qquad y = \dfrac{a}{1+t^2}$

$\dfrac{dx}{dt} = 3at^2 \qquad \dfrac{dy}{dt} = \dfrac{(1+t^2)0 - a(2t)}{(1+t^2)^2}$

$\dfrac{dy}{dt} = \dfrac{-2at}{(1+t^2)^2}$

$\dfrac{dy}{dx} = \dfrac{dy}{dt} \Big/ \dfrac{dx}{dt}$

$\dfrac{dy}{dx} = \dfrac{\dfrac{-2at}{(1+t^2)^2}}{3at^2}$

$\dfrac{dy}{dx} = \dfrac{-2}{3t(1+t^2)^2}$

At point $(a, \frac{1}{2}a)$ $\quad t = 1$

$\dfrac{dy}{dx} = \dfrac{-2}{3 \times 1 \times 2^2}$

$\dfrac{dy}{dx} = -\dfrac{1}{6}$

6) $\operatorname{cosec}^2\theta - \cot\theta = 3$

Now $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$\therefore\ 1 + \cot^2\theta - \cot\theta = 3$

$\Rightarrow \cot^2\theta - \cot\theta - 2 = 0$

$(\cot\theta + 1)(\cot\theta - 2) = 0$

Either $\cot\theta = -1$ or $\cot\theta = 2$

When $\cot\theta = -1$, $\tan\theta = -1$
$\Rightarrow \theta = 135°$

When $\cot\theta = 2$, $\tan\theta = \frac{1}{2}$
$\Rightarrow \theta = 26.6°$

Solution $\theta = 26.6°$, $\theta = 135°$

7)i) $A(1, 2, 2)$
$B(0, 0, 2)$

$\overrightarrow{AB} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix}$

line AB given by

$\underline{r} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix}$

7ii) normal is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Find angle between $\overrightarrow{AB}$ and normal

$\cos\theta = \dfrac{\begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}{\left|\begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix}\right| \left|\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right|}$

$\cos\theta = \dfrac{-1 + 0 + 0}{\sqrt{5}\sqrt{2}} = \dfrac{-1}{\sqrt{10}}$

$\theta = 108.4°$

Acute angle is therefore $71.6°$

7iii) Line BC $\underline{r} = \begin{pmatrix}0\\0\\2\end{pmatrix} + \mu\begin{pmatrix}-2\\-2\\-1\end{pmatrix}$

$$\cos\phi = \frac{\begin{pmatrix}1\\0\\1\end{pmatrix}.\begin{pmatrix}-2\\-2\\-1\end{pmatrix}}{\left|\begin{pmatrix}1\\0\\1\end{pmatrix}\right|\left|\begin{pmatrix}-2\\-2\\-1\end{pmatrix}\right|}$$

$$\cos\phi = \frac{-2+0-1}{\sqrt{2}\ \sqrt{9}} = \frac{-3}{3\sqrt{2}}$$

$$\phi = 135^\circ$$

Acute angle $\therefore$ $180-135 = 45^\circ$

7iv)

$$\sin\theta = k\sin\phi$$

$$\sin 71.6 = k\sin 45$$

$$k = \frac{\sin 71.6^\circ}{\sin 45^\circ} = 1.34$$

7v) $\underline{r} = \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0-2\mu\\0-2\mu\\2-\mu\end{pmatrix}$

Subst in plane $x+z=-1$

$$-2\mu + 2 - \mu = -1$$

$$-3\mu = -3$$

$$\mu = 1$$

Subst back in line

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-2\\-2\\1\end{pmatrix}$$

Point of intersection $(-2,-2,1)$

Distance through glass

$$= \sqrt{(-2-0)^2 + (-2-0)^2 + (1-2)^2}$$

$$= \sqrt{4+4+1} = 3\text{ cm}$$

8) i)

A) $\angle AOB = \frac{360}{12} = 30^\circ$

$\Rightarrow \angle AOC = 15^\circ$

$AC = AO\sin 15^\circ = 1\sin 15^\circ$

$AB = 2AC = 2\sin 15^\circ$

B)

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\cos 30 = 1 - 2\sin^2 15^\circ$$

$$\frac{\sqrt{3}}{2} = 1 - 2\sin^2 15^\circ$$

$$\Rightarrow 2\sin^2 15^\circ = 1 - \frac{\sqrt{3}}{2} = \frac{2-\sqrt{3}}{2}$$

$$\Rightarrow \sin^2 15^\circ = \frac{2-\sqrt{3}}{4}$$

$$\Rightarrow \sin 15^\circ = \frac{\sqrt{2-\sqrt{3}}}{2}$$

C)

Perimeter $= 12AB$

$= 24\sin 15^\circ$

$= 12\sqrt{2-\sqrt{3}}$

Circumference of circle $>$ perimeter of polygon

$\therefore$ $2\pi \times 1 > 12\sqrt{2-\sqrt{3}}$

$\Rightarrow \pi > 6\sqrt{2-\sqrt{3}}$

8ii) A) $\angle DOE = 30^\circ$

$\angle DOF = 15^\circ$

$\tan \angle DOF = \tan 15 = \frac{DF}{1}$

$\Rightarrow DF = \tan 15^\circ$

$DE = 2DF = 2\tan 15^\circ$

B) $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$\tan 30^\circ = \frac{2\tan 15^\circ}{1-\tan^2 15^\circ} = \frac{2t}{1-t^2}$

Since $\tan 30^\circ = \frac{1}{\sqrt{3}}$ we have

$\frac{1}{\sqrt{3}} = \frac{2t}{1-t^2}$

$\Rightarrow 1-t^2 = 2\sqrt{3}t$

$t^2 + 2\sqrt{3}t - 1 = 0$

c) $t = \frac{-2\sqrt{3} \pm \sqrt{12+4}}{2}$

$t = \frac{-2\sqrt{3} \pm 4}{2}$

$t = -\sqrt{3}+2$ since $t > 0$

Perimeter of polygon

$= 12\,DE = 24\tan 15$

$= 48 - 24\sqrt{3}$

Circ of circle < perimeter of polygon

$2\pi \times 1 < 48 - 24\sqrt{3}$

$\Rightarrow \pi < 24 - 12\sqrt{3}$

$\Rightarrow \pi < 12(2-\sqrt{3})$

8iii)

$6\sqrt{2-\sqrt{3}} < \pi < 12(2-\sqrt{3})$

$3.1058 < \pi < 3.2154$