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1)
$$\frac{x+1}{x^2(2x-1)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{2x-1}$$

$$x+1 \equiv Ax(2x-1) + B(2x-1) + Cx^2$$

$x=0$
$$1 = -B \qquad \Rightarrow \quad B = -1$$

$x=\frac{1}{2}$
$$\frac{3}{2} = \frac{1}{4}C \qquad \Rightarrow \quad C = 6$$

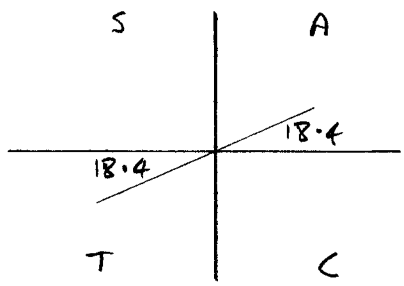
coeff of x^2
$$0 = 2A + C$$
$$0 = 2A + 6$$
$$-6 = 2A \qquad \Rightarrow \quad A = -3$$

$$\frac{x+1}{x^2(2x-1)} \equiv -\frac{3}{x} - \frac{1}{x^2} + \frac{6}{2x-1}$$

2)
$$\cot 2\theta = 3 \qquad \text{for } 0 \le \theta \le 180$$

$$\Rightarrow \tan 2\theta = \frac{1}{3}$$

$$\tan^{-1}\frac{1}{3} = 18.435$$



$$2\theta = 18.435, \ 198.435$$
$$\theta = 9.22^\circ, \ 99.22^\circ \qquad \text{to 2 d.p.}$$

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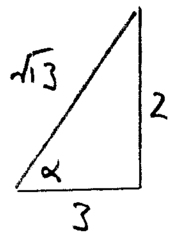
3) $3\sin x + 2\cos x$



$= \sqrt{13}\left(\frac{3}{\sqrt{13}}\sin x + \frac{2}{\sqrt{13}}\cos x\right)$

$= \sqrt{13}\sin(x+\alpha)$ where $\alpha = \tan^{-1}\frac{2}{3}$

$\alpha = 0.588$ radians

$3\sin x + 2\cos x = \sqrt{13}\sin(x + 0.588)$

Max point occurs when $x + 0.588 = \frac{\pi}{2}$

$\Rightarrow x = \frac{\pi}{2} - 0.588 = 0.983$

At this point $y = \sqrt{13}\sin\frac{\pi}{2} = \sqrt{13} = 3.606$

Max point $(0.98, 3.61)$ to 2 d.p.

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4) i) $y = \sqrt{\cos x}$

x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
y	1	0.9612	0.8409	0.6186	0

$\int_0^{\frac{\pi}{2}} \sqrt{\cos x}\,dx \approx \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3) + y_4\right]$

$= \frac{\pi}{16}\left[1 + 2(0.9612 + 0.8409 + 0.6186) + 0\right]$

≈ 1.147 to 3 dp.

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4 ii) Trapezia are all under the curve giving an under-estimate for the actual area.

More strips means trapezia get closer to curve so estimate would be larger and closer to actual area.

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5) A(2, 0, 1)

B(1, 2, 2)

C(0, −4, 1)

$$\overrightarrow{AB} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \qquad \overrightarrow{BC} = \begin{pmatrix} -1 \\ -6 \\ -1 \end{pmatrix}$$

$2\underline{i} - \underline{j} + 4\underline{k}$

$$\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = -2 - 2 + 4 = 0 \qquad \therefore \perp$$

$$\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -6 \\ -1 \end{pmatrix} = -2 + 6 - 4 = 0 \qquad \therefore \perp$$

$2\underline{i} - \underline{j} + 4\underline{k}$ is $\perp$ to two non-parallel vectors in the plane ABC and so is normal to the plane

Eqn of plane is of form

$$2x - y + 4z = d$$

(2, 0, 1) on plane so

$$4 + 4 = d$$

$$\Rightarrow \quad 8 = d$$

Plane given by $\quad 2x - y + 4z = 8$

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6) $(1+qx)^p = 1 - x + 2x^2 + \dots$

$(1+qx)^p = 1 + pqx + \frac{p.(p-1)}{1.2}(qx)^2 + \dots$

$\Rightarrow \quad pq = -1 \quad$ and $\quad \frac{p(p-1)q^2}{2} = 2$

$q = -\frac{1}{p} \qquad p(p-1)q^2 = 4$

subst for q $\qquad \frac{p(p-1)}{p^2} = 4$

$p^2 - p = 4p^2$

$0 = 3p^2 + p$

$0 = p(3p+1)$

$\Rightarrow \quad p = 0$ or $p = -\frac{1}{3}$

$p \neq 0$ since $q = -\frac{1}{p}$

$\therefore \quad p = -\frac{1}{3}$

$q = -\frac{1}{-\frac{1}{3}} = 3$

Expansion is $(1+3x)^{-\frac{1}{3}}$

Valid for $|3x| < 1$

$|x| < \frac{1}{3}$

$-\frac{1}{3} < x < \frac{1}{3}$

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7) $\underline{r_1} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4+3\lambda \\ 2 \\ 4+\lambda \end{pmatrix}$

$\underline{r_2} = \begin{pmatrix} -1 \\ 4 \\ 9 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1-\mu \\ 4+\mu \\ 9+3\mu \end{pmatrix}$

If they intersect $\begin{pmatrix} 4+3\lambda \\ 2 \\ 4+\lambda \end{pmatrix} = \begin{pmatrix} -1-\mu \\ 4+\mu \\ 9+3\mu \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$

From (2)

$2 = 4 + \mu \Rightarrow \mu = -2$

Sub in (1)

$4 + 3\lambda = -1 + 2$

$3\lambda = -1 + 2 - 4$

$3\lambda = -3$

$\lambda = -1$

Sub in (3)

$4 - 1 = 9 + 3(-2)$

$3 = 3$ ✓

Lines intersect when $\lambda = -1$, $\mu = -2$

Point of intersection $(4 + 3(-1), 2, 4 + (-1))$

$= (1, 2, 3)$

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8) $x = 2t^2$ $y = 4t$ $-\sqrt{2} \le t \le \sqrt{2}$

$P(2t^2, 4t)$

i) $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$ $\frac{dy}{dt} = 4$, $\frac{dx}{dt} = 4t$

$\frac{dy}{dx} = \frac{4}{4t} = \frac{1}{t}$

Gradient of TS = Gradient of curve at P $= \frac{1}{t}$

Gradient of TS $= \tan\theta$

$\therefore \tan\theta = \frac{1}{t}$

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ii) Gradient of QP $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{4t - 0}{2t^2 - 2} = \frac{2t}{t^2 - 1}$

$\therefore \tan\phi = \frac{2t}{t^2-1}$

Now $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{\frac{2}{t}}{1 - \frac{1}{t^2}}$ $\times \frac{t^2}{t^2}$

$= \frac{2t}{t^2 - 1} = \tan\phi$

$\therefore \tan 2\theta = \tan\phi$

$\Rightarrow \phi = 2\theta$

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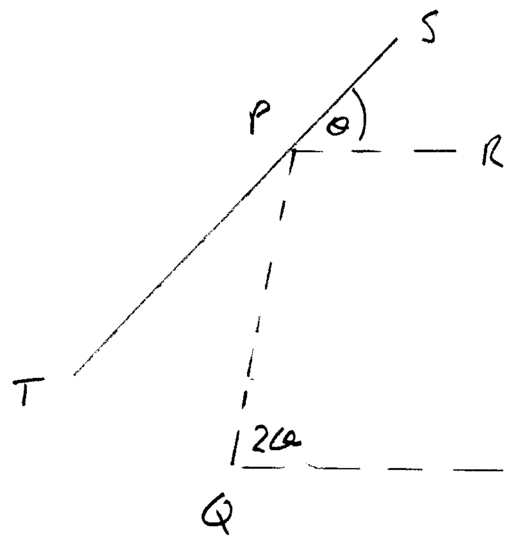
8 ii) cont



$\angle RPQ = 180 - 2\theta$ (allied angles add up to 180°)

$\angle TPQ + \angle RPQ + \angle SPR = 180°$ (angles on str line)

$\Rightarrow \angle TPQ + 180° - 2\theta + \theta = 180°$

$\Rightarrow \angle TPQ = 180° - 180° + 2\theta - \theta = \theta$

8 iii) $x = 2t^2$, $y = 4t$

$t = \frac{y}{4}$

Sub for t,

$x = 2\left(\frac{y}{4}\right)^2$

$x = \frac{y^2}{8}$

$y^2 = 8x$

When $t = \sqrt{2}$, $x = 2 \times \sqrt{2}^2 = 4$

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8 iii) cont)

$$\text{Volume} = \int_0^4 \pi y^2 \, dx$$

$$= \pi \int_0^4 8x \, dx$$

$$= \pi \left[4x^2 \right]_0^4$$

$$= 64\pi - 0$$

$$= 64\pi$$

9 i)

$$V = \pi\left(10x^2 - \frac{1}{3}x^3\right)$$

$$\frac{dV}{dx} = \pi\left(20x - x^2\right) = \pi x(20 - x)$$

Also

$$\frac{dV}{dx} = \frac{dV}{dt} \cdot \frac{dt}{dx} = \frac{dV}{dt} \Big/ \frac{dx}{dt}$$

$$\Rightarrow \frac{dV}{dx} = \frac{k(20-x)}{\frac{dx}{dt}}$$

$$\Rightarrow \frac{dx}{dt}\frac{dV}{dx} = k(20-x)$$

$$\Rightarrow \frac{dx}{dt} = \frac{k(20-x)}{\frac{dV}{dx}} = \frac{k(20-x)}{\pi x(20-x)}$$

$$\Rightarrow \frac{dx}{dt} = \frac{k}{\pi x}$$

$$\Rightarrow \pi x \frac{dx}{dt} = k$$

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9 ii)

$$\pi x \frac{dx}{dt} = k$$

$$\int \pi x \, dx = \int k \, dt$$

$$\frac{\pi x^2}{2} = kt + c$$

$x=0,\ t=0$ $\quad 0 = 0 + c \quad \Rightarrow \quad c = 0$

$$\therefore \quad \frac{\pi x^2}{2} = kt$$

Full when $x = 10$ $\quad \frac{\pi \times 10^2}{2} = kt$

$$\frac{50\pi}{k} = t$$

If full at time T $\quad T = \frac{50\pi}{k}$

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iii) $\frac{dV}{dt} = -kx$

Now $\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt}$

so $-kx = \pi x(20 - x) \frac{dx}{dt}$

$\Rightarrow \quad -k = \pi(20 - x) \frac{dx}{dt}$

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9iv)

$$\pi(20-x)\frac{dx}{dt} = -k$$

$$\int \pi(20-x)\,dx = \int -k\,dt$$

$$20\pi x - \frac{\pi x^2}{2} = -kt + c$$

$t=0,\ x=10$

$$200\pi - 50\pi = 0 + c$$

$$\Rightarrow c = 150\pi$$

$$\therefore\ 20\pi x - \frac{\pi x^2}{2} = -kt + 150\pi$$

Bowl empty when $x = 0$

$$0 = -kt + 150\pi$$

$$kt = 150\pi$$

$$t = \frac{150\pi}{k}$$

$$t = 3 \times \frac{50\pi}{k} = 3T$$