

ADVANCED GCE

MATHEMATICS (MEI)

Applications of Advanced Mathematics (C4) Paper A

4754A

Candidates answer on the Answer Booklet

OCR Supplied Materials:

- 8 page Answer Booklet
- MEI Examination Formulae and Tables (MF2)

Other Materials Required:

None

Friday 15 January 2010
Afternoon

Duration: 1 hour 30 minutes



INSTRUCTIONS TO CANDIDATES

- Write your name clearly in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully and make sure that you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [] at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

NOTE

- This paper will be followed by **Paper B: Comprehension**.

Section A (36 marks)

- 1 Find the first three terms in the binomial expansion of $\frac{1+2x}{(1-2x)^2}$ in ascending powers of x . State the set of values of x for which the expansion is valid. [7]

- 2 Show that $\cot 2\theta = \frac{1 - \tan^2 \theta}{2 \tan \theta}$.

Hence solve the equation

$$\cot 2\theta = 1 + \tan \theta \quad \text{for } 0^\circ < \theta < 360^\circ. \quad [7]$$

- 3 A curve has parametric equations

$$x = e^{2t}, \quad y = \frac{2t}{1+t}.$$

- (i) Find the gradient of the curve at the point where $t = 0$. [6]

- (ii) Find y in terms of x . [2]

- 4 The points A, B and C have coordinates $(1, 3, -2)$, $(-1, 2, -3)$ and $(0, -8, 1)$ respectively.

- (i) Find the vectors \overrightarrow{AB} and \overrightarrow{AC} . [2]

- (ii) Show that the vector $2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$ is perpendicular to the plane ABC. Hence find the equation of the plane ABC. [5]

- 5 (i) Verify that the lines $\mathbf{r} = \begin{pmatrix} -5 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ meet at the point $(1, 3, 2)$. [3]

- (ii) Find the acute angle between the lines. [4]

Section B (36 marks)

- 6 In Fig. 6, OAB is a thin bent rod, with $OA = a$ metres, $AB = b$ metres and angle $OAB = 120^\circ$. The bent rod lies in a vertical plane. OA makes an angle θ above the horizontal. The vertical height BD of B above O is h metres. The horizontal through A meets BD at C and the vertical through A meets OD at E.

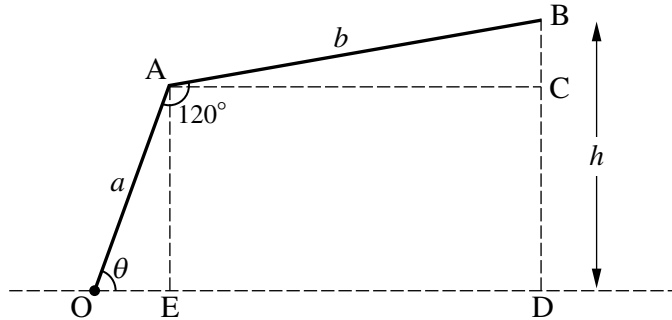


Fig. 6

- (i) Find angle BAC in terms of θ . Hence show that

$$h = a \sin \theta + b \sin(\theta - 60^\circ). \quad [3]$$

- (ii) Hence show that $h = (a + \frac{1}{2}b) \sin \theta - \frac{\sqrt{3}}{2}b \cos \theta$. [3]

The rod now rotates about O, so that θ varies. You may assume that the formulae for h in parts (i) and (ii) remain valid.

- (iii) Show that OB is horizontal when $\tan \theta = \frac{\sqrt{3}b}{2a + b}$. [3]

In the case when $a = 1$ and $b = 2$, $h = 2 \sin \theta - \sqrt{3} \cos \theta$.

- (iv) Express $2 \sin \theta - \sqrt{3} \cos \theta$ in the form $R \sin(\theta - \alpha)$. Hence, for this case, write down the maximum value of h and the corresponding value of θ . [7]

[Question 7 is printed overleaf.]

- 7 Fig. 7 illustrates the growth of a population with time. The proportion of the ultimate (long term) population is denoted by x , and the time in years by t . When $t = 0$, $x = 0.5$, and as t increases, x approaches 1.

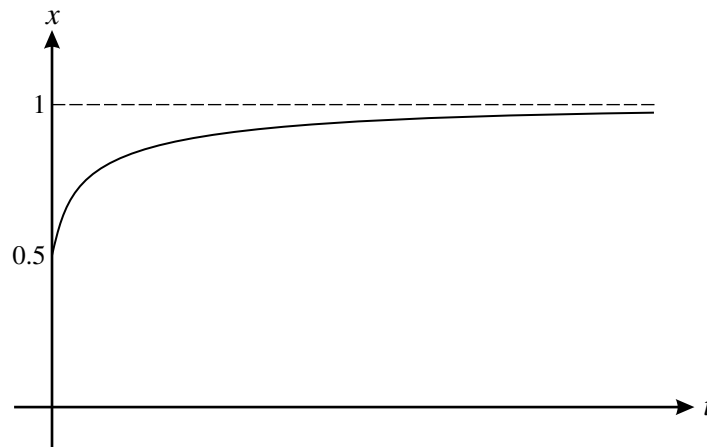


Fig. 7

One model for this situation is given by the differential equation

$$\frac{dx}{dt} = x(1 - x).$$

- (i) Verify that $x = \frac{1}{1 + e^{-t}}$ satisfies this differential equation, including the initial condition. [6]
- (ii) Find how long it will take, according to this model, for the population to reach three-quarters of its ultimate value. [3]

An alternative model for this situation is given by the differential equation

$$\frac{dx}{dt} = x^2(1 - x),$$

with $x = 0.5$ when $t = 0$ as before.

- (iii) Find constants A , B and C such that $\frac{1}{x^2(1-x)} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{1-x}$. [4]
- (iv) Hence show that $t = 2 + \ln\left(\frac{x}{1-x}\right) - \frac{1}{x}$. [5]
- (v) Find how long it will take, according to this model, for the population to reach three-quarters of its ultimate value. [2]

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ADVANCED GCE

MATHEMATICS (MEI)

Applications of Advanced Mathematics (C4) Paper B: Comprehension

4754B

Candidates answer on the Question Paper

OCR Supplied Materials:

- Insert (inserted)
- MEI Examination Formulae and Tables (MF2)

Other Materials Required:

- Rough paper

**Friday 15 January 2010
Afternoon**

Duration: Up to 1 hour



Candidate Forename		Candidate Surname	
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Centre Number						Candidate Number				
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- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully and make sure that you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- Write your answer to each question in the space provided, however additional paper may be used if necessary.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [] at the end of each question or part question.
- The insert contains the text for use with the questions.
- You may find it helpful to make notes and do some calculations as you read the passage.
- You are **not** required to hand in these notes with your question paper.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **18**.
- This document consists of **4** pages. Any blank pages are indicated.

Examiner's Use Only:	
1	
2	
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Total	

- 1 A Caesar cipher uses a shift of 11 places. Using lines 36 and 37, write down the shift for the decoding cipher in this case. [1]

.....

- 2 Using lines 143 to 146, write out the first 16 letters of the plaintext message corresponding to the ciphertext message on lines 97 to 101. [1]

.....

.....

.....

- 3 Table 4 shows an encoding cipher. Complete the table below to show part of the decoding cipher. [2]

Ciphertext	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
Plaintext					

- 4 Line 137 says ‘in string S_2 , the encoded form of the letter *A* is *N*’. Give two reasons why this is a sensible suggestion. [2]

.....

.....

.....

.....

- 5 Lines 105 and 106 say ‘Taken together, these two shifts suggest that the keyword has length 2 or 4’. Explain why this is the case. [2]

.....

.....

.....

6 Lines 107 and 108 say ‘a keyword of length 2 would form a less secure cipher than one of length 4’. Explain why this is true. [2]

.....
.....
.....

7 A long passage is encoded using the Vigenère cipher with keyword **ODE**. Write down the different ways in which the plaintext word AND could appear in the ciphertext. [2]

.....

8 A passage of plaintext is encoded by using the Caesar cipher corresponding to a shift of 2 places followed by the Vigenère cipher with keyword **ODE**.

(i) The first letter in the plaintext passage is F. Show that the first letter in the transmitted text is V. [1]

.....
.....

(ii) The first four letters in the transmitted text are *VFIU*. What are the first four letters in the plaintext passage? [3]

.....
.....
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.....

(iii) The 800th letter in the transmitted text is *W*. What is the 800th letter in the plaintext passage? [2]

.....
.....
.....
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THERE ARE NO QUESTIONS PRINTED ON THIS PAGE.



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**ADVANCED GCE
MATHEMATICS (MEI)**

Applications of Advanced Mathematics (C4) Paper B: Comprehension
INSERT

4754B

**Friday 15 January 2010
Afternoon**

Duration: Up to 1 hour



INSTRUCTIONS TO CANDIDATES

- This insert contains the text for use with the questions.

INFORMATION FOR CANDIDATES

- This document consists of **8** pages. Any blank pages are indicated.

Cipher Systems

Introduction

Imagine you want to send a written message to a friend containing confidential information. You are keen to ensure that the information remains private but you are aware that other people might try to intercept the message before it reaches its destination. Therefore it is necessary to use some form of code, or *cipher*, known to you and your friend, so that anybody who intercepts the message will find it difficult to extract the information.

5

You use an encoding cipher to encode your message (called the *plaintext*) and then transmit the resulting message (called the *ciphertext*). Your friend will then decode the message using an appropriate decoding cipher. This *cipher system* is illustrated in Fig. 1.

10

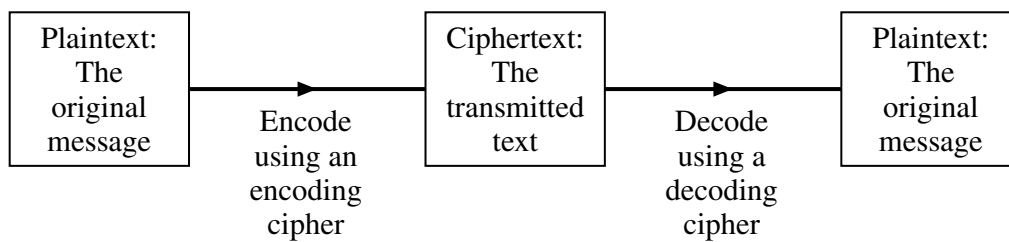


Fig. 1

What is the most appropriate cipher system to use so that you can send and receive messages quickly and cheaply, whilst minimising the risk that an interceptor will be able to decode the ciphertext?

This has been an important question for at least two thousand years, since messages were written on the shaved heads of messengers who would then wait until their hair grew back before setting off on journeys to the intended recipients. More sophisticated cipher systems have been developed and some of these are described in this article. The methods are not necessarily representative of those used in practice today, but highlight some of the weaknesses which need to be avoided when trying to design a secure cipher system.

15

Caesar cipher

This is the most basic form of cipher. In a Caesar cipher, each letter in the message is replaced by the letter a fixed number of places further on in the alphabet to give the text to transmit.

20

For example, Table 2 shows the Caesar cipher corresponding to a shift of the alphabet by 21 places.

Plaintext	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
Ciphertext	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U

Table 2

Using this encoding cipher, the plaintext

IF I HAVE SEEN FARTHER THAN OTHER MEN,
IT IS BECAUSE I HAVE STOOD ON THE SHOULDERS OF GIANTS.

25

would be transmitted as the following ciphertext.

DA D CVQZ NZZI AVMOCZM OCVI JOCZM HZI,
DO DN WZXVPNZ D CVQZ NOJJY JI OCZ NCJPGYZMN JA BDVION.

[Warning Do not spend time in this examination checking the accuracy of this or any other ciphertext.]

30

To disguise the lengths of words, it is common for ciphertext to be transmitted in blocks of fixed length, with punctuation removed. For example, the ciphertext above could be transmitted as follows.

DADCV QZNZZ IAVMO CZMOC VIJOC ZMHZI DODNW ZXVPN
ZDCVQ ZNOJJ YJIOC ZNCJP GYZMN JABDV ION

35

Notice that the decoding cipher is a shift of the alphabet in the same direction by 5 places. This is shown in Table 3.

Ciphertext	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
Plaintext	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E

Table 3

Clearly such a cipher system is not at all secure as there are only 25 shifts of the alphabet which an interceptor would have to try to be certain of extracting the message.

Substitution cipher

40

If the sender and receiver agree in advance on a letter substitution, then a more secure cipher system can be used. Table 4 shows an example of a more secure encoding cipher.

Plaintext	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
Ciphertext	Q	R	F	O	H	Z	K	B	D	M	T	V	A	S	I	W	E	Y	L	N	G	X	P	C	J	U

Table 4

It is unlikely that the sender or receiver would be able to remember the 26 letter substitutions and so it would be necessary to keep a written copy of the cipher. However, by writing it down there is a risk of it being discovered by an interceptor.

45

An alternative is simply to use a phrase, with repeated letters removed, to form the first few letters of the cipher. The letters not appearing in the phrase are then used in alphabetical order.

For example, removing repeated letters from the phrase

GOD WROTE THE UNIVERSE IN THE LANGUAGE OF MATHEMATICS

leaves

GODWRTEHUNIVSLAFMC

50

and this gives the ciphertext of the first 18 letters of the alphabet. The corresponding encoding cipher is given in Table 5.

Plaintext	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
Ciphertext	G	O	D	W	R	T	E	H	U	N	I	V	S	L	A	F	M	C	B	J	K	P	Q	X	Y	Z

Table 5

If a ciphertext message based on a cipher of this type is intercepted, and the interceptor has reason to believe that letter substitution has been used, then the number of possible arrangements of the alphabet that could have been used is

55

$$26! = 403\,291\,461\,126\,605\,635\,584\,000\,000.$$

If the interceptor had the computing power to check 1 000 000 arrangements per second, whilst also checking the resulting messages to see if any were meaningful, then it would still take longer than the age of the universe to check all possibilities.

60

However, the interceptor might still have a chance of extracting the message. If the original message was in English and it was sufficiently long (about 200 letters is usually enough), then there are several techniques the interceptor might use.

- If the sender had simply encoded the words and left the gaps between the words, then it would not be difficult to discover the encoded forms of the letters A and I. This is one reason why it is normal for coded messages to be transmitted in fixed-length blocks.
- By far the most common three-letter word in English text is 'the' and the frequent appearance of a three-letter string in the ciphertext would probably give the encoded forms of T, H and E.
- In English, the letter Q is always followed by the letter U. If, in the ciphertext, one letter is always followed by the same other letter, this might give the interceptor the encoded forms of Q and U. This is one extreme example of a relationship between letters but other relationships also exist.
- Fig. 6 shows the percentage frequencies of the 26 letters in a large sample of English text from a wide variety of sources. In a sufficiently long passage of ciphertext, it would be sensible to look at those letters occurring most frequently and investigate if they could represent E, T and A. Once these letters are decoded, knowledge of the English language would quickly reveal many complete words.

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75

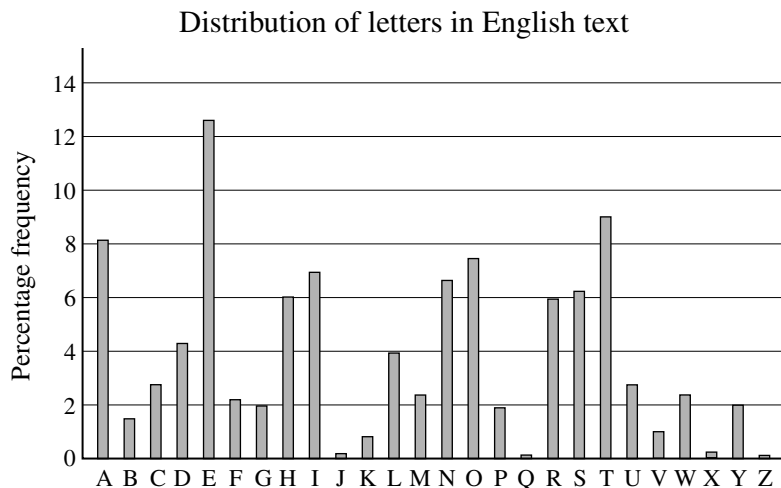


Fig. 6

Vigenère Cipher

In the ciphers mentioned above, a letter in ciphertext will always represent the same letter in the plaintext message. This is not the case with a Vigenère cipher. This cipher is illustrated using the following example.

80

First select a *keyword*. In the following example, the keyword is **ODE**. Rotations of the 26 letters of the alphabet are then written under the alphabet, starting with each of the letters in the keyword, as shown in Table 7.

Plaintext	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
O	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N
D	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C
E	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D

Table 7

To encode a message, each of these rows is used in turn. For example, to encode the word NEWTON you would encode

- the first N using row **O**,
- the E using row **D**,
- the W using row **E**,
- the T using row **O**, and so on.

85

90

Therefore the word NEWTON would be transmitted in ciphertext as *BHAHRR*. You will notice that the two Ns in NEWTON appear as different letters in the ciphertext and the *Hs* and *Rs* in the ciphertext do not correspond to repeated letters in the plaintext.

This cipher is more secure than the others described so far but, for a sufficiently long passage of ciphertext, it is still vulnerable to frequency analysis, especially if the keyword is short. For example, consider the following ciphertext (in which the keyword used is not **ODE**).

95

NUMFU GPXGN BBWVI GMCIM NRZGM YQDYG PXJNQ GNRZL
 IEBAY CWXNF UNMGJ XVRIN NVNNF GPXCQ MTMYQ DYGPX
 WBTHO EAHLG **PXQBZ** WMZCL NSQMN BOXNU **MKCAI** AUEUH
 HVWNM JIRVR INNLO LNUMY CEAMN RAMNU **MKYVA** GICMK
 GNVXH GXEUP MBHGP **XQBZE** XSWKO TTRGN BAYZI MCPA

100

The string **GPXQBZ** appears twice, one a shift of 84 places from the other. It is reasonable to assume that in both cases the same six-letter string in the plaintext has been encoded. If this is indeed the case then the length of the keyword must be a factor of 84. There are also two occurrences of the string **NUMK**, one a shift of 40 places from the other. Taken together, these two shifts suggest that the keyword has length 2 or 4.

105

Assume that the keyword is of length 4, since a keyword of length 2 would form a less secure cipher than one of length 4.

To carry out a frequency analysis, the ciphertext is split into four strings, S_1 , S_2 , S_3 and S_4 , each made up of every fourth letter of the original ciphertext.

110

S_1 : NUGWMNMYJNIYNMVFNCMYWOLQMNNNCUHMVNNCNNYIGHUHQXOGYC
 S_2 : UGNVCRYGNRECFGRVGYGBEGBZSBUAJEVJRLUERUVCNGPGBSTNZP
 S_3 : MPBIIIZQPQZBWUJINPMQPTAPZCQOMIUWIIQMAAMAMVXMPZWTBIA
 S_4 : FXBGMGDGLAXNXNNXTDXHHXWLMXKAHNRNL YMMKGKXEBXEKRAM

These text strings may be analysed using the information in Fig. 6. (The analysis may not always be successful with strings as short as these.)

115

Notice the following two characteristics of the distribution shown in Fig. 6.

- The three highest frequencies correspond to E, T and A.
- After the peak at T there is a run of six letters with low frequencies; this is the longest such run.

120

Compare this distribution with the distribution of letters in the first text string, S_1 , shown in Fig. 8 below.

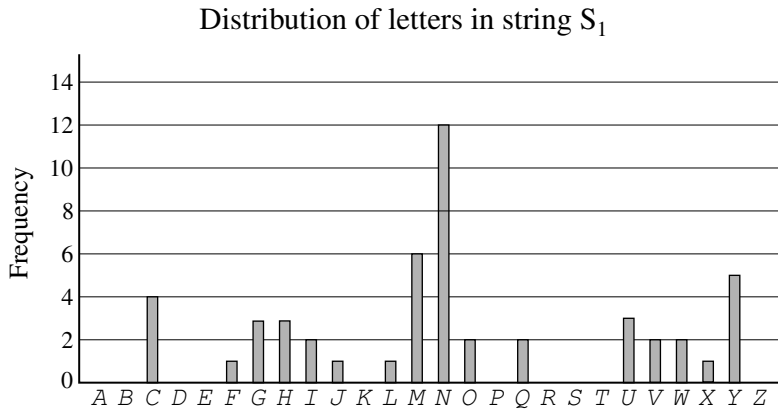


Fig. 8

There are two reasons to suspect that the letter N in string S_1 is the encoded form of the letter T .

- N in Fig. 8 is followed by a run of six letters with low frequency.
- The corresponding encoded forms of A and E would be U and Y respectively; both of these occur quite frequently.

125

These two reasons suggest that it is worth looking more closely at this particular shift.

Starting the distribution at U , as shown in Fig. 9, allows a direct comparison to be made with the distribution in Fig. 6.

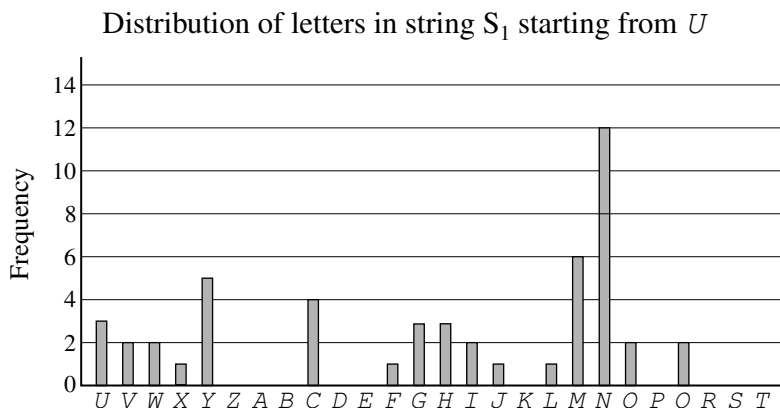


Fig. 9

The string is made up of only 50 letters so a very close match with Fig. 6 is not to be expected. However the distributions are sufficiently similar to accept this shift for now and investigate the other three strings; it is only after all four strings have been decoded that we can determine if the shifts are correct.

130

The frequency distributions corresponding to the strings S_2 , S_3 and S_4 are shown in Figs. 10, 11 and 12 respectively.

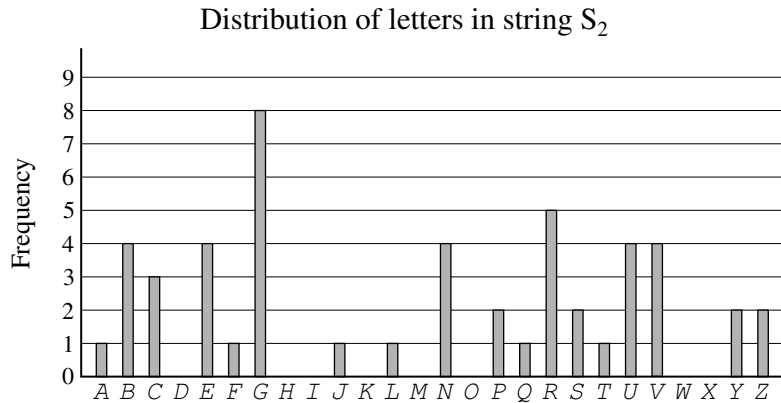


Fig. 10

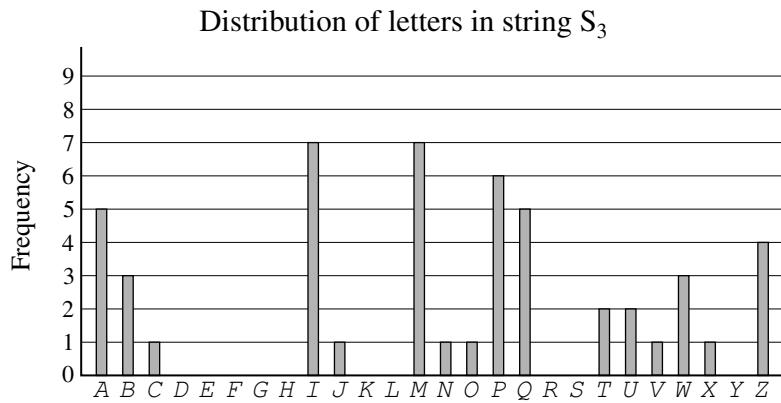


Fig. 11

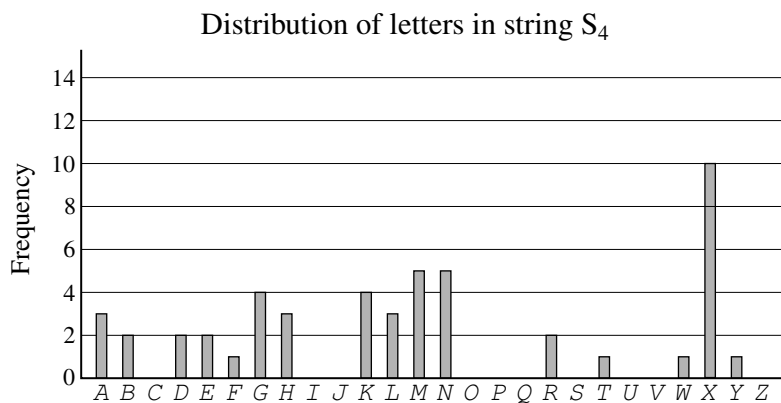


Fig. 12

By employing reasoning similar to that used in the analysis of string S_1 , it is likely that

- in string S_2 , the encoded form of the letter A is N,
- in string S_3 , the encoded form of the letter A is I.

For string S_4 it is not quite so clear. However, it may be the case that the keyword used is a word in the English language. So it is sensible to check, in string S_4 , if the encoded form of the letter A is T since this would make the four-letter keyword **UNIT**. With this keyword, the strings S_1 , S_2 , S_3 and S_4 are decoded to give the following strings. 140

T_1 : TAMCSTSEPTOETSBLISECURWSTTTIANSBTTITTEOMNANWDUMEI
 T_2 : HTAIPELTAERPSTEITDLTORTOMFOHNRIWEYHREHIPATCTOFGAMC
 T_3 : EHTAARIHIRTOMBFAFHEIHLSHRUIGEAMOAAIESSESENPEHROLTAS 145
 T_4 : MEINTNKENSHEUEUEAKEOOEDSTERHOUYUSFTTRNRELIELRYHT

This does indeed result in a meaningful plaintext message.

In conclusion

The methods considered have all involved substituting each letter with another letter. Variations on these are possible, such as having an encoded form of each of the 676 two-letter pairs (such as AA, AB, AC, ...) or encoding a message using one cipher and then encoding the resulting ciphertext using another cipher. 150

The science of cryptography affects all our lives. Every time you send an email, use a cashpoint machine or make purchases on the internet, the information you transmit is encoded so that only your intended recipient can read it. 155

Government intelligence is heavily dependent upon the ability to transmit information securely whilst also trying to break the ciphers used by others. Indeed, it is estimated that the Second World War was shortened by two years, thereby saving many lives, thanks to the intelligence gained by the mathematicians working in cryptography at Bletchley Park.

As computing power becomes more sophisticated, more secure codes are continually being sought. The most secure codes in use today rely heavily on techniques from pure mathematics. 160



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4754 (C4) Applications of Advanced Mathematics

1		$\frac{1+2x}{(1-2x)^2} = (1+2x)(1-2x)^{-2}$ $= (1+2x)\left[1 + (-2)(-2x) + \frac{(-2)(-3)}{1.2}(-2x)^2 + \dots\right]$ $= (1+2x)[1 + 4x + 12x^2 + \dots]$ $= 1 + 4x + 12x^2 + 2x + 8x^2 + \dots$ $= 1 + 6x + 20x^2 + \dots$ <p>Valid for $-1 < -2x < 1$ $\Rightarrow -\frac{1}{2} < x < \frac{1}{2}$</p>	M1 A1 A1 M1 A1 M1 A1 [7]	binomial expansion power -2 unsimplified, correct sufficient terms
2		$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$ $\Rightarrow \cot 2\theta = \frac{1}{\tan 2\theta} = \frac{1 - \tan^2 \theta}{2 \tan \theta} *$ $\cot 2\theta = 1 + \tan \theta$ $\Rightarrow \frac{1 - \tan^2 \theta}{2 \tan \theta} = 1 + \tan \theta$ $\Rightarrow 1 - \tan^2 \theta = 2 \tan \theta + 2 \tan^2 \theta$ $\Rightarrow 3 \tan^2 \theta + 2 \tan \theta - 1 = 0$ $\Rightarrow (3 \tan \theta - 1)(\tan \theta + 1) = 0$ $\Rightarrow \tan \theta = 1/3, \theta = 18.43^\circ, 198.43^\circ$ $\text{or } \tan \theta = -1, \theta = 135^\circ, 315^\circ$	M1 E1 M1 M1 A3,2,1, 0 [7]	oe eg converting either side into a one line fraction(s) involving $\sin \theta$ and $\cos \theta$. quadratic = 0 factorising or solving 18.43°, 198.43°, 135°, 315° -1 extra solutions in the range
3	(i)	$\frac{dy}{dt} = \frac{(1+t).2 - 2t.1}{(1+t)^2} = \frac{2}{(1+t)^2}$ $\frac{dx}{dt} = 2e^{2t}$ $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ $\Rightarrow \frac{dy}{dx} = \frac{\frac{2}{(1+t)^2}}{2e^{2t}} = \frac{1}{e^{2t}(1+t)^2}$ $t = 0 \Rightarrow dy/dx = 1$	M1A1 B1 M1 A1 B1ft [6]	

	(ii)	$2t = \ln x \Rightarrow t = \frac{1}{2} \ln x$ $\Rightarrow y = \frac{\ln x}{1 + \frac{1}{2} \ln x} = \frac{2 \ln x}{2 + \ln x}$	M1 A1 [2]	or t in terms of y
4	(i)	$\overline{AB} = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}, \overline{AC} = \begin{pmatrix} -1 \\ -11 \\ 3 \end{pmatrix}$	B1 B1 [2]	
	(ii)	$\mathbf{n} \cdot \overline{AB} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} = -4 + 1 + 3 = 0$ $\mathbf{n} \cdot \overline{AC} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -11 \\ 3 \end{pmatrix} = -2 + 11 - 9 = 0$ $\Rightarrow \text{plane is } 2x - y - 3z = d$ $x = 1, y = 3, z = -2 \Rightarrow d = 2 - 3 + 6 = 5$ $\Rightarrow \text{plane is } 2x - y - 3z = 5$	M1 E1 E1 M1 A1 [5]	scalar product
5	(i)	$x = -5 + 3\lambda = 1 \Rightarrow \lambda = 2$ $y = 3 + 2 \times 0 = 3$ $z = 4 - 2 = 2, \text{ so } (1, 3, 2) \text{ lies on 1st line.}$ $x = -1 + 2\mu = 1 \Rightarrow \mu = 1$ $y = 4 - 1 = 3$ $z = 2 + 0 = 2, \text{ so } (1, 3, 2) \text{ lies on 2}^{\text{nd}} \text{ line.}$	M1 E1 E1 [3]	finding λ or μ verifying two other coordinates for line 1 verifying two other coordinates for line 2
	(ii)	<p>Angle between $\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$</p> $\cos \theta = \frac{3 \times 2 + 0 \times (-1) + (-1) \times 0}{\sqrt{10} \sqrt{5}}$ $= 0.8485 \dots$ $\Rightarrow \theta = 31.9^\circ$	M1 M1 A1 A1 [4]	direction vectors only allow M1 for any vectors or 0.558 radians

6	(i)	$\begin{aligned} \text{BAC} &= 120 - 90 - (90 - \theta) \\ &= \theta - 60 \\ \Rightarrow \text{BC} &= b \sin(\theta - 60) \\ \text{CD} &= \text{AE} = a \sin \theta \\ \Rightarrow h &= \text{BC} + \text{CD} = a \sin \theta + b \sin(\theta - 60^\circ) * \end{aligned}$	B1 M1 E1 [3]	
	(ii)	$\begin{aligned} h &= a \sin \theta + b \sin(\theta - 60^\circ) \\ &= a \sin \theta + b(\sin \theta \cos 60 - \cos \theta \sin 60) \\ &= a \sin \theta + \frac{1}{2} b \sin \theta - \frac{\sqrt{3}}{2} b \cos \theta \\ &= \left(a + \frac{1}{2} b\right) \sin \theta - \frac{\sqrt{3}}{2} b \cos \theta * \end{aligned}$	M1 M1 E1 [3]	corr compound angle formula $\sin 60 = \sqrt{3}/2$, $\cos 60 = 1/2$ used
	(iii)	$\begin{aligned} \text{OB horizontal when } h &= 0 \\ \Rightarrow \left(a + \frac{1}{2} b\right) \sin \theta - \frac{\sqrt{3}}{2} b \cos \theta &= 0 \\ \Rightarrow \left(a + \frac{1}{2} b\right) \sin \theta &= \frac{\sqrt{3}}{2} b \cos \theta \\ \Rightarrow \frac{\sin \theta}{\cos \theta} &= \frac{\frac{\sqrt{3}}{2} b}{a + \frac{1}{2} b} \\ \Rightarrow \tan \theta &= \frac{\sqrt{3} b}{2a + b} * \end{aligned}$	M1 M1 E1 [3]	$\frac{\sin \theta}{\cos \theta} = \tan \theta$
	(iv)	$\begin{aligned} 2 \sin \theta - \sqrt{3} \cos \theta &= R \sin(\theta - \alpha) \\ &= R(\sin \theta \cos \alpha - \cos \theta \sin \alpha) \\ \Rightarrow R \cos \alpha &= 2, R \sin \alpha = \sqrt{3} \\ \Rightarrow R^2 &= 2^2 + (\sqrt{3})^2 = 7, R = \sqrt{7} = 2.646 \text{ m} \\ \tan \alpha &= \sqrt{3}/2, \alpha = 40.9^\circ \\ \text{So } h &= \sqrt{7} \sin(\theta - 40.9^\circ) \\ \Rightarrow h_{\max} &= \sqrt{7} = 2.646 \text{ m} \\ &\text{when } \theta - 40.9^\circ = 90^\circ \\ \Rightarrow \theta &= 130.9^\circ \end{aligned}$	M1 B1 M1A1 B1ft M1 A1 [7]	

7	(i)	$\frac{dx}{dt} = -1(1+e^{-t})^{-2} \cdot -e^{-t}$ $= \frac{e^{-t}}{(1+e^{-t})^2}$ $1-x = 1 - \frac{1}{1+e^{-t}}$ $1-x = \frac{1+e^{-t}-1}{1+e^{-t}} = \frac{e^{-t}}{1+e^{-t}}$ $\Rightarrow x(1-x) = \frac{1}{1+e^{-t}} \frac{e^{-t}}{1+e^{-t}} = \frac{e^{-t}}{(1+e^{-t})^2}$ $\Rightarrow \frac{dx}{dt} = x(1-x)$ <p>When $t = 0$, $x = \frac{1}{1+e^0} = 0.5$</p>	M1 A1 M1 A1 E1 B1 [6]	chain rule substituting for $x(1-x)$ $1-x = \frac{1+e^{-t}-1}{1+e^{-t}} = \frac{e^{-t}}{1+e^{-t}}$ [OR, M1 A1 for solving differential equation for t , B1 use of initial condition, M1 A1 making x the subject, E1 required form]
	(ii)	$\frac{1}{(1+e^{-t})} = \frac{3}{4}$ $\Rightarrow e^{-t} = 1/3$ $\Rightarrow t = -\ln 1/3 = 1.10 \text{ years}$	M1 M1 A1 [3]	correct log rules
	(iii)	$\frac{1}{x^2(1-x)} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{1-x}$ $\Rightarrow 1 = A(1-x) + Bx(1-x) + Cx^2$ $x = 0 \Rightarrow A = 1$ $x = 1 \Rightarrow C = 1$ $\text{coefft of } x^2: 0 = -B + C \Rightarrow B = 1$	M1 M1 B(2,1,0) [4]	clearing fractions substituting or equating coeffs for A,B or C $A = 1, B = 1, C = 1$ www
	(iv)	$\int \frac{dx}{x^2(1-x)} = \int dt$ $\Rightarrow t = \int \left(\frac{1}{x^2} + \frac{1}{x} + \frac{1}{1-x} \right) dx$ $= -1/x + \ln x - \ln(1-x) + c$ <p>When $t = 0$, $x = 1/2 \Rightarrow 0 = -2 + \ln 1/2 - \ln 1/2 + c$</p> $\Rightarrow c = 2.$ $\Rightarrow t = -1/x + \ln x - \ln(1-x) + 2$ $= 2 + \ln \frac{x}{1-x} - \frac{1}{x} *$	M1 B1 B1 M1 E1 [5]	separating variables $-1/x + \dots$ $\ln x - \ln(1-x)$ ft their A,B,C substituting initial conditions
	(v)	$t = 2 + \ln \frac{3/4}{1-3/4} - \frac{1}{3/4} = \ln 3 + \frac{2}{3} = 1.77 \text{ yrs}$	M1A1 [2]	

1	15	B1	
2	THE MATHEMATICIAN	B1	
3	M H X I Q 3 or 4 correct – award 1 mark	B2	
4	Two from Ciphertext N has high frequency E would then correspond to ciphertext R which also has high frequency T would then correspond to ciphertext G which also has high frequency A is preceded by a string of six letters displaying low frequency	B1 B1	oe oe
5	The length of the keyword is a factor of both 84 and 40. The <u>only</u> common factors of 84 and 40 are 1,2 and 4 (and a keyword of length 1 can be dismissed in this context)	M1 E1	
6	Longer strings to analyse so letter frequency more transparent. Or there are fewer 2-letter keywords to check	B2	
7	OQH DRR EBG One or two accurate – award 1 mark	B2	
8 (i) (ii) (iii)	Evidence of intermediate H FACE Evidence of intermediate HCEG – award 2 marks Evidence of accurate application of one of the two decoding ciphers - award 1 mark $800 = (3 \times 266) + 2$; the second row gives T so plaintext is R	B1 B3 M1 A1	 Use of second row

4754 Applications of Advanced Mathematics (C4)

General Comments

This paper was similar in standard to the January 2009 paper. It provided questions that were accessible to all candidates. As usual the entry for the January paper, although increasing, was relatively low. The standard of work was, once again, high. Good scores were obtained by many and some clear presentation was seen. The Comprehension proved to be well understood by all and scored highly. Section A of Paper A provided a wide range of accessible questions. The first part of question 7 in Section B provided the greatest challenge; here many solved the differential equation rather than verifying (as the question asked).

Candidates should be advised to read questions carefully. Many lost unnecessary marks by failing to 'verify' or 'show' fully when required. These words are part of the question and should not be overlooked. Similarly, 'explain' needs words and 'hence' should be seen as a big hint - even though alternative methods will sometimes gain credit. Questions requiring answers in degrees or radians are often answered in the alternative form or for the wrong range. Candidates could often avoid losing marks in these cases if they took more care.

Comments on Individual Questions

Paper A

Section A

- 1 The binomial expansion was well answered. There were few errors other than numerical slips such as using $2x$ instead of $-2x$. The most commonly lost marks were in the validity statement which was often omitted.
- 2 A few candidates failed to obtain the E mark in the first part by failing to state both $\cot 2\theta = 1/\tan 2\theta$ and the double angle formula for $\tan 2\theta$ even though one felt that they meant to write these down.
The second part was almost always successful. Most candidates found and solved the quadratic equation and found all four solutions.
- 3 Here dx/dt was sometimes seen with the coefficient $\frac{1}{2}$ instead of 2. dy/dt was usually correct if approached as a product or quotient, but many missed this and gave $dy/dt = -2t/(1+t)^2$ having only differentiated the denominator. The general method was usually correct but not all could accurately substitute $t = 0$.
In the second part, most approached from x rather than y . Generally $t = \frac{1}{2} \ln x$ was obtained, but substitution and simplification caused errors when finding y in terms of x .
- 4 Part (i) was almost always correct. Some gave the answer in co-ordinate form which was condoned on this occasion.
In part (ii) there were many completely correct solutions. However, a number failed to realise it was necessary to prove that the normal was perpendicular to two vectors in the plane in order to establish that it was a normal vector.
For the equation of the plane, common errors included using $i/j/k$ instead of $x/y/z$ or obtaining -5 or 7 instead of 5.

- 5 Whilst most candidates correctly found λ or μ or both, some failed to verify that these values worked for all the co-ordinates in both lines.
The angle between the lines was usually found correctly although a few candidates used the position vectors instead of the direction vectors.

Section B

- 6
- (i) Angle BAC was usually found successfully but a few omitted this or made incorrect or unclear statements.
 - (ii) This was almost always correct. Most wrote down the correct formula and substituted for $\cos 60$ and $\sin 60$. A few lost the b when expanding the bracket.
 - (iii) For those that put $h=0$ and divided by $\cos \theta$ this proved very successful. Some omitted this part. Occasionally incorrect statements such as $\sin \theta = \sqrt{3}b$, $\cos \theta = 2a+b$ were seen. Others failed to substitute $h = 0$ but continued with the correct method on the RHS without changing the LHS. Full marks could be obtained by substituting $\tan \theta = \sqrt{3}b/(2a+b)$ and working backwards to show that $h = 0$.
 - (iv) This was usually successful. Some candidates attempted only this part of question 6. Some lost a mark here for working in radians.
- 7
- (i) This proved to be the most difficult part of the paper. The majority chose to solve the differential equation rather than to use verification as asked in the question. Those who chose to verify were usually successful and some efficient solutions were seen. The approach from the differential equation was prone to error. Whilst some good solutions were seen, common errors included omitting the constant of integration, integrating $1/(1-x)$ as $+\ln(1-x)$, using incorrect log rules (particularly with the constant) and when rearranging to make x the subject. The mark for checking the initial condition was usually obtained. Verification proved to be the easiest route and it is unclear why so few tried this.
 - (ii) This part was usually well done although some used values of x other than 0.75 ; 0.875 was the most common. Others failed to manipulate their equation correctly and found that they needed the logarithm of a negative number and others prematurely approximated their answer.
 - (iii) The partial fractions were well done by most but a surprising number equated the coefficients of x^2 as $0 = B + C$ instead of $0 = -B + C$ and so incorrectly obtained $B = -1$. Others started with $1 = Ax(1-x) + Bx^2(1-x) + Cx^3$ and lost marks - in some cases for apparently correct values of the coefficients found from incorrect working.
 - (iv) As the answer to this part was given, there was some 'fiddling' of answers from those that failed to integrate $1/(1-x)$ as $-\ln(1-x)$. Some failed to show the separation of the variables and did not show $t =$, only working with the right hand side, and many also tried to integrate $1/x^2$ as a logarithm. The general method was usually known and most tried to find the constant of integration even following incorrect working.
 - (v) This was usually correct although some did substitute incorrect values of x .

Paper B
The Comprehension

- 1 This was usually correct although some wrote out the alphabet. Occasionally 5 or 11 were given instead of 15.
- 2 '*The mathematician*' was almost always given.
- 3 *M H X I Q* was usually correct.
- 4 Most candidates understood what was required here. A variety of acceptable reasons were given although some were not sufficiently precise or were unclear.
- 5 The majority of candidates realised that the values given were factors of 84 and 40. However some failed to make it clear that these were the **only** common factors. In some cases the words 'factor' and 'multiple' were confused.
- 6 Few fully understood what was required here but many obtained some credit for their answers. Many were unclear. Few realised that with a keyword of length 2 the strings would be longer and so the frequency analysis would make the letter frequency more obvious.
- 7 The correct answers were usually seen.
- 8
 - (i) There was usually evidence of the intermediate H although some encoded in the wrong order.
 - (ii) FACE was usually correct.
 - (iii) Some candidates miscalculated which row was needed. Others decoded by doing the ciphers in the wrong order. However, Row D followed by plaintext T and then plaintext R was usually correctly given.