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1)
$$\frac{1}{(2x+1)(x^2+1)}$$

$$\equiv \frac{A}{2x+1} + \frac{Bx+C}{x^2+1}$$

$$1 \equiv A(x^2+1) + (Bx+C)(2x+1)$$

When $x = -\frac{1}{2}$

$$1 = A\left(\left(-\tfrac{1}{2}\right)^2+1\right) = A \times \frac{5}{4}$$

$$\Rightarrow \quad A = \frac{4}{5}$$

when $x = 0$

$$1 = A(1) + (C)(1)$$

$$1 - A = C$$

$$1 - \frac{4}{5} = C$$

$$\Rightarrow \quad \frac{1}{5} = C$$

Coeff of x^2

$$0 = A + 2B$$

$$-A = 2B$$

$$-\frac{A}{2} = B$$

$$\Rightarrow \quad B = -\frac{2}{5}$$

$$\therefore \quad \frac{1}{(2x+1)(x^2+1)}$$

$$= \frac{4}{5(2x+1)} + \frac{1-2x}{5(x^2+1)}$$

2)
$$\sqrt[3]{1+3x}$$

$$= (1+3x)^{\frac{1}{3}}$$

$$= 1 + \tfrac{1}{3}(3x) + \frac{\frac{1}{3}\cdot -\frac{2}{3}}{1.2}(3x)^2 + \dots$$

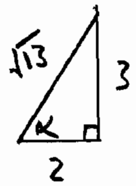
$$= 1 + x - x^2 + \dots$$

Valid for $|3x| < 1$

$$|x| < \frac{1}{3}$$

$$-\frac{1}{3} < x < \frac{1}{3}$$

3)
$$2\sin\theta - 3\cos\theta$$



$$= \sqrt{13}\left(\frac{2}{\sqrt{13}}\sin\theta - \frac{3}{\sqrt{13}}\cos\theta\right)$$

$$= \sqrt{13}\left(\sin(\theta - \alpha)\right)$$

where
$$\alpha = \tan^{-1}\frac{3}{2} = 0.983$$

$$= \sqrt{13}\sin(\theta - 0.983)$$

$$1 + 2\sin\theta - 3\cos\theta$$

$$= 1 + \sqrt{13}\sin(\theta - 0.983)$$

Max value $= 1 + \sqrt{13}$

Min value $= 1 - \sqrt{13}$

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4) $x = 2\sin\theta$, $y = \cos 2\theta$

i) $\frac{dx}{d\theta} = 2\cos\theta$ $\quad \frac{dy}{d\theta} = -2\sin 2\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{-2\sin 2\theta}{2\cos\theta}$$

$$\frac{dy}{dx} = -\frac{\sin 2\theta}{\cos\theta}$$

$$= -\frac{2\sin\theta\cos\theta}{\cos\theta}$$

$$\frac{dy}{dx} = -2\sin\theta$$

When $\theta = \frac{\pi}{3}$, $\frac{dy}{dx} = -2\sin\frac{\pi}{3}$

$$= -2 \times \frac{\sqrt{3}}{2}$$

$$\frac{dy}{dx} = -\sqrt{3}$$

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ii) $y = \cos 2\theta = 1 - 2\sin^2\theta$

$x = 2\sin\theta$

$\Rightarrow \frac{x}{2} = \sin\theta$

$\therefore y = 1 - 2\left(\frac{x}{2}\right)^2$

$$y = 1 - \frac{x^2}{2}$$

4 i) cont)

$x = 2\sin\theta$, $y = \cos 2\theta$

when $\theta = \frac{\pi}{3}$

$x = 2\sin\frac{\pi}{3} = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$

$y = \cos\frac{2\pi}{3} = -\frac{1}{2}$

Point is $\left(\sqrt{3}, -\frac{1}{2}\right)$

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5) $\operatorname{cosec}^2\theta = 1 + 2\cot\theta$

for $-180 \le \theta \le 180$

$1 + \cot^2\theta = 1 + 2\cot\theta$

$\cot^2\theta - 2\cot\theta = 0$

$\cot\theta(\cot\theta - 2) = 0$

Either $\cot\theta = 0$

$\Rightarrow \tan\theta = \infty$ $\quad \theta = 90^\circ$ or $\theta = -90^\circ$

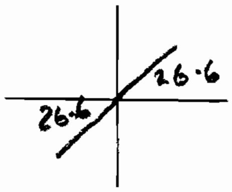
or $\cot\theta - 2 = 0$

$\cot\theta = 2$

$\tan\theta = \frac{1}{2}$

$\theta = \tan^{-1}\frac{1}{2} = 26.6^\circ$

or -153.4°



$\theta = -153.4^\circ, -90^\circ, 26.6^\circ, 90^\circ$

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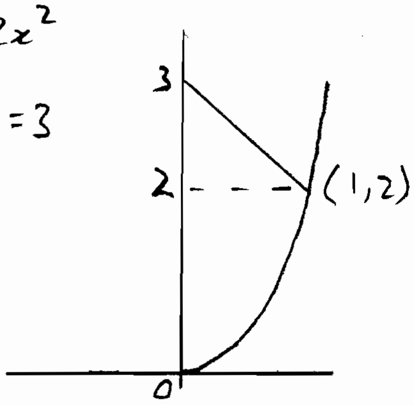
6) $y = 2x^2$

$x + y = 3$



Volume of top part (cone)

$= \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times 1^2 \times 1$

$= \frac{\pi}{3}$

Volume of bottom part

$= \pi \int_0^2 x^2 \, dy$

$= \pi \int_0^2 \frac{y}{2} \, dy$

$= \left[\frac{\pi y^2}{4} \right]_0^2$

$= \pi - 0 = \pi$

Total volume

$= \pi + \frac{\pi}{3} = \frac{4\pi}{3}$ units^3

Section B

7) $A(2, 0, 3)$
$B(-2, 0, 1)$
$C(0, 4, 4)$
$D(-2, 4, k)$

i)

$\vec{AB} = \begin{pmatrix} -4 \\ 0 \\ -2 \end{pmatrix}$ $\vec{AC} = \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}$

$\cos \angle BAC = \frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}||\vec{AC}|}$

$= \frac{\begin{pmatrix} -4 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}}{\left|\begin{pmatrix} -4 \\ 0 \\ -2 \end{pmatrix}\right| \left|\begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}\right|}$

$= \frac{8 + 0 - 2}{\sqrt{20}\sqrt{21}} = \frac{6}{\sqrt{420}}$

$\angle ABC = \cos^{-1}\left(\frac{6}{\sqrt{420}}\right)$

$= 73.0^\circ$

ii) $x + y - 2z + d = 0$

$A(2, 0, 3)$

$2 + 0 - 6 + d = 0$

$\Rightarrow d = 4$

for A to be on this plane

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$x + y - 2z + 4 = 0$

$B(-2, 0, 1)$

$-2 + 0 - 2 + 4 = 0$ ✓

$C(0, 4, 4)$

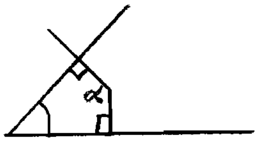
$0 + 4 - 8 + 4 = 0$ ✓

A, B, C all on this plane

so Plane ABC given by

$x + y - 2z + 4 = 0$

Angle between plane and horizontal plane



same as acute angle between normals

ABC normal $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

Horizontal normal $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\cos\alpha = \frac{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{\left|\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}\right|\left|\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right|}$$

$$\cos\alpha = \frac{-2}{\sqrt{6}\sqrt{1}} = -\frac{2}{\sqrt{6}}$$

$\alpha = 144.7°$

Acute angle $= 180 - 144.7$

$= 35.3°$

iii) $D(-2, 4, k)$ is on

$x + y - 2z + 4 = 0$

$-2 + 4 - 2k + 4 = 0$

$6 = 2k$

$3 = k$

$\therefore \quad k = 3$

$$\vec{CD} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -4 \\ 0 \\ -2 \end{pmatrix} = \frac{1}{2}\vec{AB}$$

$\therefore$ $\vec{CD}$ and $\vec{AB}$ are parallel

$\vec{BD} = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}$ $\quad \vec{AC}$ $\begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}$

These are not parallel so

ABCD is a trapezium

Since $\vec{CD} = \frac{1}{2}\vec{AB}$

$CD : AB = 1 : 2$

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8)

$V = \frac{1}{3}x^3 \qquad \frac{dV}{dt} = -kx$

i)

$\frac{dx}{dt} = \frac{dx}{dV} \cdot \frac{dV}{dt}$

$= \frac{1}{\frac{dV}{dx}} \times \frac{dV}{dt}$

Now $\frac{dV}{dx} = \frac{3x^2}{3} = x^2$

so $\frac{dx}{dt} = \frac{1}{x^2} \times -kx$

$\frac{dx}{dt} = -\frac{k}{x}$

$\Rightarrow \quad x\frac{dx}{dt} = -k$

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ii)

$x\frac{dx}{dt} = -k$

$\Rightarrow \quad \int x\,dx = \int -k\,dt$

$\Rightarrow \quad \frac{x^2}{2} = -kt + c$

Given $x = 10$ when $t = 0$

$\Rightarrow \quad \frac{10^2}{2} = c$

$\Rightarrow \quad c = 50$

$\Rightarrow \quad \frac{x^2}{2} = -kt + 50$

$\Rightarrow \quad x^2 = -2kt + 100$

$\Rightarrow \quad x = \sqrt{100 - 2kt}$

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iii) when $t = 50$, $x = 0$

$0 = \sqrt{100 - 100k}$

$\Rightarrow \quad k = 1$

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iv)

Now $\frac{dV}{dt} = -kx + 1$

but $k = 1$

so $\frac{dV}{dt} = 1 - x$

$\frac{dx}{dt} = \frac{1}{\frac{dV}{dx}} \times \frac{dV}{dt}$

$= \frac{1}{x^2}(1-x)$

$= \frac{1-x}{x^2}$

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v) $\frac{1}{1-x} - x - 1$

$= \frac{1}{1-x} - (1+x)$

$= \frac{1 - (1+x)(1-x)}{1-x}$

$= \frac{1 - (1^2 - x^2)}{1-x}$

8v) cont)

$$= \frac{1 - 1 + x^2}{1-x} = \frac{x^2}{1-x}$$

$$\frac{dx}{dt} = \frac{1-x}{x^2}$$

$$\int \frac{x^2}{1-x}\,dx = \int 1\,dt$$

$$\int \left(\frac{1}{1-x} - x - 1 \right) dx = \int 1\,dt$$

$$-\ln|1-x| - \frac{x^2}{2} - x = t + c$$

when $x = 0$, $t = 0$

$$-\ln 1 - 0 - 0 = 0 + c$$

$$\Rightarrow c = 0$$

$$\therefore\ -\ln|1-x| - \frac{x^2}{2} - x = t$$

$$\Rightarrow\ \ln\left(\frac{1}{1-x}\right) - \frac{x^2}{2} - x = t$$

as required

vi) As $x \to 1$

$$t \to \ln(\infty) - \frac{1}{2} - 1 = \infty$$

$\therefore$ x never reaches 1