MEI CORE 4 JUNE 2012

1.) $\dfrac{4x}{x+1} - \dfrac{3}{2x+1} = 1$

$4x(2x+1) - 3(x+1)$

$= (x+1)(2x+1)$

$8x^2 + 4x - 3x - 3 = 2x^2 + 2x + x + 1$

$8x^2 + x - 3 = 2x^2 + 3x + 1$

$8x^2 + x - 3 - 2x^2 - 3x - 1 = 0$

$6x^2 - 2x - 4 = 0$

$3x^2 - x - 2 = 0$

$(3x+2)(x-1) = 0$

$\Rightarrow 3x + 2 = 0 \Rightarrow x = -\frac{2}{3}$

or $x - 1 = 0 \Rightarrow x = 1$

Solution $x = -\frac{2}{3}$, $x = 1$

2) $\sqrt{1+2x} = (1+2x)^{\frac{1}{2}}$

$= 1 + \frac{1}{2}(2x) + \dfrac{\frac{1}{2}\cdot -\frac{1}{2}}{1.2}(2x)^2$

$+ \dfrac{\frac{1}{2}\cdot -\frac{1}{2}\cdot -\frac{3}{2}}{1.2.3}(2x)^3 + +$

$= 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - + \ldots$

valid for $|2x| < 1$

$|x| < \frac{1}{2}$

$-\frac{1}{2} < x < \frac{1}{2}$

3.i) $\dfrac{dV}{dt} = k\sqrt{V}$

If $V = (\frac{1}{2}kt + c)^2$

$\dfrac{dV}{dt} = 2(\frac{1}{2}kt + c)(\frac{1}{2}k)$

$= k(\frac{1}{2}kt + c)$

$= k\sqrt{V}$

$\therefore$ differential equation is satisfied

3 ii) $t = 1$, $V = 10{,}000$
$t = 2$, $V = 40{,}000$

Substitute

$10000 = (\frac{1}{2}k + c)^2$ (1)

$40000 = (k + c)^2$ (2)

From (1) $100 = \frac{1}{2}k + c$ (3)

From (2) $200 = k + c$ (4)

(4) − (3) $100 = \frac{1}{2}k$

$\Rightarrow k = 200$

$\Rightarrow c = 0$

$\therefore V = (\frac{1}{2} \times 200t + 0)^2$

$V = 10000t^2$

4.) $\sec^2\theta + \operatorname{cosec}^2\theta$

$= \dfrac{1}{\cos^2\theta} + \dfrac{1}{\sin^2\theta}$

MEI CORE 4 JUNE 2012

4 cont) $= \dfrac{\sin^2\theta + \cos^2\theta}{\cos^2\theta \sin^2\theta}$

$= \dfrac{1}{\cos^2\theta \sin^2\theta}$

$= \sec^2\theta \operatorname{cosec}^2\theta$

5.) $\sin(x + 45°) = 2\cos x$

$\sin x \cos 45° + \cos x \sin 45° = 2\cos x$

$\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x = 2\cos x$

$\sin x + \cos x = 2\sqrt{2} \cos x$

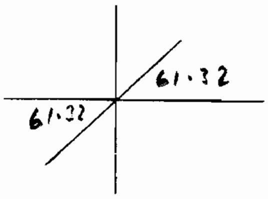
$\Rightarrow \sin x = 2\sqrt{2}\cos x - \cos x$

$\Rightarrow \sin x = (2\sqrt{2} - 1)\cos x$

$\Rightarrow \dfrac{\sin x}{\cos x} = \tan x = 2\sqrt{2} - 1$

$\Rightarrow x = \tan^{-1}(2\sqrt{2} - 1)$

$x = 61.32°$




or $x = 241.32°$

6) $\dfrac{dy}{dx} = \dfrac{y}{x(x+1)}$

$\displaystyle\int \frac{1}{y}\, dy = \int \frac{1}{x(x+1)}\, dx$

$\displaystyle\int \frac{1}{y}\, dy = \int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$

$\ln y = \ln x - \ln(x+1) + c$

$x = 1, y = 1$

$\ln 1 = \ln 1 - \ln 2 + c$

$\Rightarrow c = \ln 2$

$\therefore \ln y = \ln x - \ln(x+1) + \ln 2$

$\ln y = \ln\left(\dfrac{2x}{x+1}\right)$

$\Rightarrow y = \dfrac{2x}{x+1}$

7.) i) $x = 2\cos\theta$, $y = \sin 2\theta$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\theta = -\frac{\pi}{2}$ $x = 2\cos\left(-\frac{\pi}{2}\right) = 0$

$y = \sin(-\pi) = 0$

So $O(0,0)$ when $\theta = -\frac{\pi}{2}$

when $\theta = 0$

$x = 2\cos 0 = 2$

$y = 2\sin 0 = 0$

so $P(2,0)$ when $\theta = 0$

when $\theta = \frac{\pi}{2}$ $x = 2\cos\frac{\pi}{2} = 0$

$y = \sin\pi = 0$

so $O(0,0)$ when $\theta = \frac{\pi}{2}$

MEI CORE 4 JUNE 2012

7ii) $\dfrac{dx}{d\theta} = -2\sin\theta, \quad \dfrac{dy}{d\theta} = 2\cos 2\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}$$

$$\frac{dy}{dx} = \frac{2\cos 2\theta}{-2\sin\theta} = -\frac{\cos 2\theta}{\sin\theta}$$

When $\theta = \frac{\pi}{2}$, $\dfrac{dy}{dx} = -\dfrac{\cos\pi}{\sin\frac{\pi}{2}}$

$$= 1$$

When $\theta = -\frac{\pi}{2}$ $\dfrac{dy}{dx} = \dfrac{-\cos(-\pi)}{\sin(-\frac{\pi}{2})}$

$$= \frac{--1}{-1} = -1$$

tangents $\perp$ since $-1 \times 1 = -1$

7iii) At Q $\dfrac{dy}{dx} = 0$

$\Rightarrow \cos 2\theta = 0$

$\Rightarrow 2\theta = \frac{\pi}{2}$ or $-\frac{\pi}{2}$

$\Rightarrow \theta = \pm\frac{\pi}{4}$

When $\theta = \frac{\pi}{4}$, $x = 2\cos\frac{\pi}{4} = \frac{2}{\sqrt{2}}$

$y = \sin\frac{\pi}{2} = 1$

so $Q(\sqrt{2}, 1)$

Note $\theta = -\frac{\pi}{4}$ would give

$R(\sqrt{2}, -1)$

7iv) $x = 2\cos\theta$

$\Rightarrow \dfrac{x}{2} = \cos\theta$

$\Rightarrow \dfrac{x^2}{4} = \cos^2\theta$

$\Rightarrow \dfrac{x^2}{4} = 1 - \sin^2\theta$

$\Rightarrow \sin^2\theta = 1 - \dfrac{x^2}{4}$

Now $y = \sin 2\theta = 2\sin\theta\cos\theta$

$\Rightarrow y^2 = 4\sin^2\theta\cos^2\theta$

$\Rightarrow y^2 = 4\left(1 - \frac{x^2}{4}\right)\left(\frac{x^2}{4}\right)$

$\Rightarrow y^2 = x^2\left(1 - \frac{x^2}{4}\right)$

7v) $V = \displaystyle\int_0^2 \pi y^2\,dx$

$$= \pi\int_0^2 x^2\left(1 - \frac{x^2}{4}\right)dx$$

$$= \pi\int_0^2 \left(x^2 - \frac{x^4}{4}\right)dx$$

$$= \pi\left[\frac{x^3}{3} - \frac{x^5}{20}\right]_0^2$$

$$= \pi\left[\left(\frac{8}{3} - \frac{32}{20}\right) - 0\right]$$

$$= \pi\left[\frac{160-96}{60}\right] = \frac{64}{60}\pi$$

$= \dfrac{16}{15}\pi$ units$^3$

or $3.35$ units$^3$ to 3 s.f.

MEI CORE 4 JUNE 2012

8) i) $A(1,2,4)$
$A'(2,4,1)$

$$AA' = \begin{pmatrix} 2-1 \\ 4-2 \\ 1-4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

which is $\perp$ to $x+2y-3z=0$

(components match coefficients of $x, y, z$)

ii) AB $\underline{r} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+\lambda \\ 2-\lambda \\ 4+2\lambda \end{pmatrix}$$

Sub in plane

$$1+\lambda+2(2-\lambda)-3(4+2\lambda)=0$$

$$1+\lambda+4-2\lambda-12-6\lambda=0$$

$$-7-7\lambda=0$$

$$-7=7\lambda \quad \Rightarrow \quad \lambda=-1$$

Sub in line

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-1 \\ 2+1 \\ 4-2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$$

$B(0,3,2)$

iii) $A'(2,4,1)$
$B(0,3,2)$
$A(1,2,4)$

$$\overrightarrow{A'B} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

$$\overrightarrow{BA} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\cos\theta = \frac{\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}}{\left|\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}\right| \left|\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\right|}$$

$$\cos\theta = \frac{-2+1+2}{\sqrt{6}\sqrt{6}} = \frac{1}{6}$$

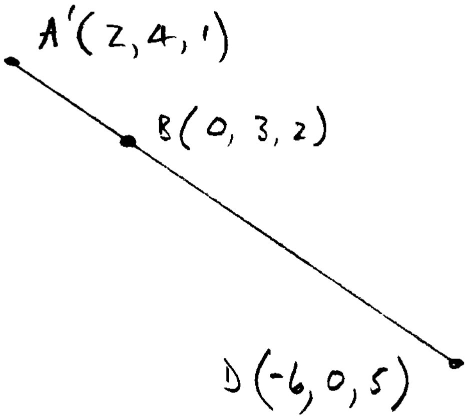
$$\theta = \cos^{-1}\left(\tfrac{1}{6}\right) = 80.4^\circ$$

iv) BC crosses $Oxz$ when $y=0$

If it crosses at D then

similar $\Delta$s can be used to

locate D




Note that when moving from

B to D the y coord changes

3 times as much as from A' to B

So x and z coords do likewise