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1) $\dfrac{x}{(1+x)(1-2x)}$

i) $\equiv \dfrac{A}{(1+x)} + \dfrac{B}{(1-2x)}$

$\Rightarrow \quad x \equiv A(1-2x) + B(1+x)$

$x = -1,$

$-1 = A(1-2(-1))$

$-1 = 3A$

$\Rightarrow \quad A = -\frac{1}{3}$

$x = \frac{1}{2},$

$\frac{1}{2} = B(1+\frac{1}{2}) = \frac{3B}{2}$

$\Rightarrow \quad B = \frac{1}{3}$

$\therefore \quad \dfrac{x}{(1+x)(1-2x)}$

$\equiv \dfrac{1}{3(1-2x)} - \dfrac{1}{3(1+x)}$

ii) $\dfrac{x}{(1+x)(1-2x)}$

$= \frac{1}{3}\left[\dfrac{1}{1-2x} - \dfrac{1}{1+x}\right]$

$= \frac{1}{3}\left[(1-2x)^{-1} - (1+x)^{-1}\right]$

$\approx \frac{1}{3}\left[\left[1 + -1(-2x) + \frac{-1.-2}{1.2}(-2x)^2 ..\right] - \left[1 + -1x + \frac{-1.-2}{1.2}x^2 ...\right]\right]$

$= \frac{1}{3}\left[\left[1 + 2x + 4x^2 ..\right] - \left[1 - x + x^2 ...\right]\right]$

$= \frac{1}{3}\left[3x + 3x^2 ...\right]$

$= x + x^2 + ...$

To be valid $|x| < 1$

and $|2x| < 1$

$\Rightarrow \quad |x| < 1$

$\Rightarrow \quad -1 < x < 1$

2) $\operatorname{cosec} x + 5\cot x = 3\sin x$

$\Rightarrow \quad \dfrac{1}{\sin x} + \dfrac{5\cos x}{\sin x} = 3\sin x$

$\Rightarrow \quad 1 + 5\cos x = 3\sin^2 x$

$\Rightarrow \quad 1 + 5\cos x = 3(1 - \cos^2 x)$

$\Rightarrow \quad 1 + 5\cos x = 3 - 3\cos^2 x$

$\Rightarrow \quad 3\cos^2 x + 5\cos x - 2 = 0$

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2 cont)

$$(3\cos x - 1)(\cos x + 2) = 0$$

$\Rightarrow \quad 3\cos x - 1 = 0$

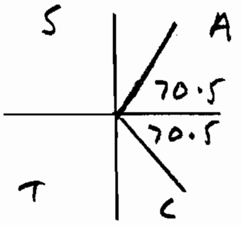
$\cos x = \frac{1}{3}$

or $\quad \cos x + 2 = 0$

$\cos x = -2$ impossible

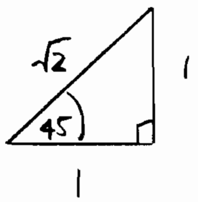
when $\cos x = \frac{1}{3}$

$x = \cos^{-1}\frac{1}{3} = 70.5°$



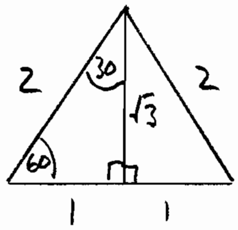
$x = 70.5°, \; 289.5°$

3)



$\tan 45° = \frac{1}{1}$

$\tan 45° = 1$



$\tan 30° = \frac{1}{\sqrt{3}}$

$\tan 75° = \tan(45° + 30°)$

$$= \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1 \times \frac{1}{\sqrt{3}}}$$

(Multiply top and bottom by $\sqrt{3}$)

$$= \frac{\sqrt{3} + 1}{\sqrt{3} - 1}$$

$$= \frac{\sqrt{3} + 1}{\sqrt{3} - 1} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1}$$

$$= \frac{3 + \sqrt{3} + \sqrt{3} + 1}{3 - 1}$$

$$= \frac{4 + 2\sqrt{3}}{2}$$

$$= 2 + \sqrt{3}$$

4) $A(0, 1, 3)$

i) $B(-2, 2, 5)$

$$\overrightarrow{AB} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

$$\underline{r} = \overrightarrow{OA} + \lambda \overrightarrow{AB}$$

Line l given by

$$\underline{r} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

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4 ii) $\underline{r} = \begin{pmatrix} 0 - 2\lambda \\ 1 + \lambda \\ 3 + 2\lambda \end{pmatrix}$

Plane

$$x + 3y + 2z = 4$$

Subst in plane

$$-2\lambda + 3(1+\lambda) + 2(3+2\lambda) = 4$$

$$-2\lambda + 3 + 3\lambda + 6 + 4\lambda = 4$$

$$5\lambda + 9 = 4$$

$$5\lambda = -5$$

$$\lambda = -1$$

Subst in line

$$\underline{r} = \begin{pmatrix} -2(-1) \\ 1 - 1 \\ 3 - 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

Point of intersection is

$$(2, 0, 1)$$

4 iii) Normal to plane is $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

$$\cos\theta = \frac{\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}}{\left|\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}\right| \left|\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}\right|}$$

$$\cos\theta = \frac{-2 + 3 + 4}{\sqrt{1^2+3^2+2^2}\,\sqrt{(-2)^2+1^2+2^2}}$$

$$\cos\theta = \frac{5}{\sqrt{14}\,\sqrt{9}} = \frac{5}{\sqrt{126}}$$

$$\theta = \cos^{-1}\left(\frac{5}{\sqrt{126}}\right)$$

$$\theta = 63.5^\circ$$

5) $A(3, 2, -1)$

$B(-1, 1, 2)$

$C(10, 5, -5)$

If $\overrightarrow{OC} = \lambda \overrightarrow{OA} + \mu \overrightarrow{OB}$.

$$\begin{pmatrix} 10 \\ 5 \\ -5 \end{pmatrix} = \lambda \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$\Rightarrow \quad 10 = 3\lambda - \mu \quad (1)$

and $\quad 5 = 2\lambda + \mu \quad (2)$

(1)+(2) $\quad 15 = 5\lambda$

$\Rightarrow \lambda = 3$

Sub in (2) $\quad 5 = 6 + \mu$

$\Rightarrow \mu = -1$

Test to see if these values for λ, μ satisfy z-coordinate

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5 cont) We require

$$-5 = -\lambda + 2\mu$$

when $\lambda = 3$, $\mu = -1$

$$-5 = -3 + 2(-1)$$

$$-5 = -5 \quad \checkmark$$

$$\therefore \quad \overrightarrow{OC} = 3\overrightarrow{OA} - 1\overrightarrow{OB}$$

The fact that $\overrightarrow{OC}$ can be expressed as a combination of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ implies that O, A, B, C all lie in the same plane.

Section B

6) i)

$$v\frac{dv}{dx} + 4x = 0$$

(Given $v = 4$ when $x = 1$)

$$v\frac{dv}{dx} = -4x$$

$$\int v\,dv = \int -4x\,dx$$

$$\frac{v^2}{2} = -2x^2 + c$$

Sub $v = 4$, $x = 1$

$$\frac{16}{2} = -2(1)^2 + c$$

$$8 = -2 + c$$

$$10 = c$$

$$\therefore \quad \frac{v^2}{2} = -2x^2 + 10$$

$$\Rightarrow \quad v^2 = -4x^2 + 20$$

ii) $x = \cos 2t + 2\sin 2t$

When $t = 0$

$$x = \cos 0 + 2\sin 0$$

$$x = 1 + 0 = 1 \quad \checkmark$$

$$\frac{dx}{dt} = -2\sin 2t + 4\cos 2t$$

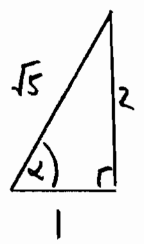
When $t = 0$

$$v = \frac{dx}{dt} = 2\sin 0 + 4\cos 0$$

$$= 0 + 4 = 4$$

$$\therefore \quad v = 4$$

iii) $x = R\cos(2t - \alpha)$



$$x = \sqrt{5}\left(\frac{1}{\sqrt{5}}\cos 2t + \frac{2}{\sqrt{5}}\sin 2t\right)$$

$$x = \sqrt{5}\cos(2t - \alpha)$$

where $\alpha = \tan^{-1}\frac{2}{1} = 1.107$ radians

$$x = \sqrt{5}\cos(2t - 1.107)$$

$$\Rightarrow \quad v = -2\sqrt{5}\sin(2t - 1.107)$$

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6iii) cont)

In this case

$$v^2 = (-2\sqrt{5})^2 \sin^2(2t - 1.107)$$

$$= 20 \sin^2(2t - 1.107)$$

and

$$x^2 = 5\cos^2(2t - 1.107)$$

so $4x^2 = 20\cos^2(2t - 1.107)$

$$\therefore v^2 + 4x^2 = 20\left(\sin^2(2t - 1.107) + \cos^2(2t - 1.107)\right)$$

$$\Rightarrow v^2 + 4x^2 = 20$$

so $v^2 = 20 - 4x^2$

6iv)

$$x = \sqrt{5}\cos(2t - 1.107)$$

Max value of x is $\sqrt{5}$

This first occurs when

$$2t - 1.107 = 0$$

$$2t = 1.107$$

$$t = 0.5535 \text{ s}$$

$$t = 0.554 \text{ s} \quad \text{to 3 s.f.}$$

7)

$$x = 5\ln u$$

$$y = u + \frac{1}{u}$$

$$(1 \le u \le 10)$$

i) At C, $u = 10$

So $C\left(5\ln 10, 10 + \frac{1}{10}\right)$

$C(5\ln 10, 10.1)$

$$\Rightarrow |OA| = 5\ln 10$$

$$|AC| = 10.1$$

At B $u = 1$ so $B\left(0, 1 + \frac{1}{1}\right)$

$B(0, 2)$

$$\therefore |OB| = 2$$

ii)

$$\frac{dy}{dx} = \frac{dy}{du} \Big/ \frac{dx}{du}$$

$$\frac{dx}{du} = \frac{5}{u} \qquad \frac{dy}{du} = 1 - \frac{1}{u^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1 - \frac{1}{u^2}}{\frac{5}{u}}$$

$$= \frac{u^2 - 1}{5u}$$

(This was obtained by multiplying top and bottom by u^2)

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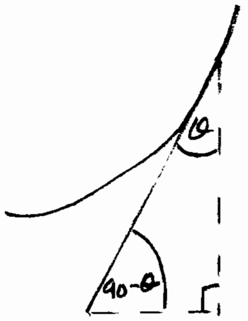
7 ii cont) At C where $u = 10$

$$\frac{dy}{dx} = \frac{10^2 - 1}{5 \times 10} = \frac{99}{50}$$



$$\tan(90-\theta) = \frac{99}{50}$$

$$90-\theta = \tan^{-1}\left(\frac{99}{50}\right)$$

$$90-\theta = 63.2^\circ$$

$$\Rightarrow \theta = 26.8^\circ$$

iii) If $x = 5\ln u$

$$\frac{x}{5} = \ln u$$

$$\Rightarrow u = e^{\frac{x}{5}}$$

$$\therefore y = u + \frac{1}{u}$$

$$y = e^{\frac{x}{5}} + \frac{1}{e^{\frac{x}{5}}}$$

$$y = e^{\frac{x}{5}} + e^{-\frac{x}{5}}$$

iv)

$$V = \pi \int_0^{5\ln 10} y^2\, dx$$

$$V = \pi \int_0^{5\ln 10} \left(e^{\frac{x}{5}} + e^{-\frac{x}{5}}\right)^2 dx$$

$$= \pi \int_0^{5\ln 10} \left(e^{\frac{2x}{5}} + 2 + e^{-\frac{2x}{5}}\right) dx$$

$$= \pi \left[\frac{e^{\frac{2x}{5}}}{\frac{2}{5}} + 2x - \frac{e^{-\frac{2x}{5}}}{\frac{2}{5}} \right]_0^{5\ln 10}$$

$$= \pi \left[\frac{5}{2}e^{\frac{2x}{5}} + 2x - \frac{5}{2}e^{-\frac{2x}{5}} \right]_0^{5\ln 10}$$

$$= \pi \left[\left(\frac{5}{2}e^{2\ln 10} + 10\ln 10 - \frac{5}{2}e^{-2\ln 10} \right) - \left(\frac{5}{2} + 0 - \frac{5}{2} \right) \right]$$

$$= \pi \left[\frac{5}{2}e^{\ln 100} + 10\ln 10 - \frac{5}{2}e^{-\ln 100} \right]$$

$$= \pi \left[\frac{500}{2} + 10\ln 10 - \frac{5}{200} \right]$$

$$= 857.657$$

$$= 858 \text{ units}^3 \text{ to } 3 \text{ s.f.}$$