MEI CORE 4 JUNE 2014

1) $$\frac{3x}{(2-x)(4+x^2)} \equiv \frac{A}{2-x} + \frac{Bx+C}{4+x^2}$$

$$3x \equiv A(4+x^2) + (Bx+C)(2-x)$$

$x=2$

$$6 = A(4+2^2)$$

$$6 = 8A \quad \Rightarrow \quad A = \frac{3}{4}$$

$x=0$ $$0 = 4A + 2C$$

$$0 = 3 + 2C \quad \Rightarrow \quad C = -\frac{3}{2}$$

Coeff of x^2

$$0 = A - B$$

$$B = A \quad \Rightarrow \quad B = \frac{3}{4}$$

$$\frac{3x}{(2-x)(4+x^2)} \equiv \frac{3}{4(2-x)} + \frac{3x-6}{4(4+x^2)}$$

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2) $$(4+x)^{3/2} = \left(4\left(1+\frac{x}{4}\right)\right)^{3/2} = 4^{3/2}\left(1+\frac{x}{4}\right)^{3/2}$$

$$= 8\left[1 + \frac{3}{2}\left(\frac{x}{4}\right) + \frac{\frac{3}{2}\cdot\frac{1}{2}}{1.2}\left(\frac{x}{4}\right)^2 + + \dots\right]$$

$$= 8\left[1 + \frac{3x}{8} + \frac{3x^2}{128} + + \dots\right]$$

$$= 8 + 3x + \frac{3x^2}{16} + + \dots$$

Valid for $\left|\frac{x}{4}\right| < 1$

$$-4 < x < 4$$

MEI CORE 4 JUNE 2014

3)

$$y = x^3 + \sqrt{\sin x} \qquad 0 \le x \le \frac{\pi}{4}$$

i)

x	y
0	0
$\pi/16$	0.4493
$\pi/8$	0.6792
$3\pi/16$	0.9498
$\pi/4$	1.3254

$$A \approx \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3) + y_4\right]$$

$$A \approx \frac{\pi}{32}\left[0 + 2\left(0.4493 + 0.6792 + 0.9498\right) + 1.3254\right]$$

$$A \approx 0.5382$$

$$A \approx 0.538 \quad \text{to } 3 \text{ d.p.}$$

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ii) Not possible to say

Trapezia close to $x = 0$ will be under curve whilst trapezia close to $x = \frac{\pi}{4}$ will be partially above curve. These errors will cancel each other out to some extent, but it is not predictable to say whether estimate would increase or decrease.

MEI CORE 4 JUNE 2014

4) i) Show $\cos(\alpha+\beta) = \dfrac{1-\tan\alpha\tan\beta}{\sec\alpha\sec\beta}$

$$\frac{1-\tan\alpha\tan\beta}{\sec\alpha\sec\beta} = \left(1 - \frac{\sin\alpha}{\cos\alpha}\frac{\sin\beta}{\cos\beta}\right)\cos\alpha\cos\beta$$

$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$= \cos(\alpha+\beta)$$

ii)

$$\cos 2\alpha = \cos(\alpha+\alpha) = \frac{1-\tan\alpha\tan\alpha}{\sec^2\alpha}$$

$$= \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$$

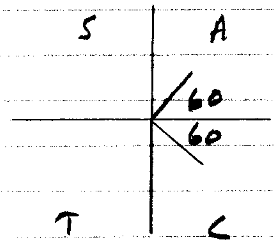
iii)

$$\frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1}{2} \qquad 0 \le \theta \le 180^\circ$$

$$\Rightarrow \cos 2\theta = \frac{1}{2} \qquad \cos^{-1}\left(\tfrac{1}{2}\right) = 60^\circ$$

$$\Rightarrow 2\theta = 60^\circ, 300^\circ$$

$$\Rightarrow \theta = 30^\circ, 150^\circ$$



Alternatively, $\dfrac{1-\tan^2\theta}{1+\tan^2\theta} = \dfrac{1}{2} \Rightarrow$

$$2(1-\tan^2\theta) = 1+\tan^2\theta$$

$$2 - 2\tan^2\theta = 1+\tan^2\theta$$

$$1 = 3\tan^2\theta$$

$$\frac{1}{3} = \tan^2\theta$$

$$\pm\frac{1}{\sqrt{3}} = \tan\theta$$

$$\theta = 30^\circ, 150^\circ$$

MEI CORE 4 JUNE 2014

5) $x = e^{3t} \quad y = te^{2t}$

i)
$$\frac{dx}{dt} = 3e^{3t}, \quad \frac{dy}{dt} = 2te^{2t} + e^{2t} = e^{2t}(2t+1)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{e^{2t}(2t+1)}{3e^{3t}} = \frac{2t+1}{3e^{t}}$$

When $t = 1$, gradient $= \frac{dy}{dx} = \frac{2(1)+1}{3e^{1}} = \frac{3}{3e} = \frac{1}{e}$

ii) $x = e^{3t}$

$\Rightarrow \ln x = 3t$

$\Rightarrow t = \frac{1}{3}\ln x$

Subst for t in $y = te^{2t}$

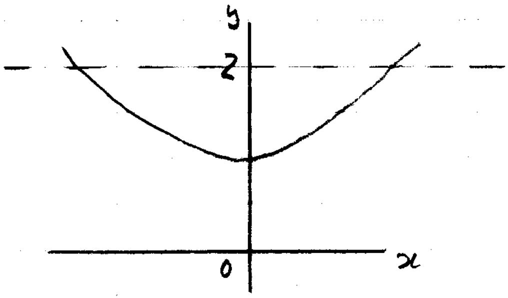
$$y = \frac{1}{3}\ln x \, e^{\frac{2}{3}\ln x}$$

$$y = \frac{1}{3}\ln x \, e^{\ln x^{2/3}}$$

$$y = \frac{1}{3}\ln x \times x^{2/3}$$

$$y = \frac{1}{3}x^{2/3}\ln x$$

6) $y = (1+2x^2)^{\frac{1}{3}}$ $\qquad y = 2$



When $x = 0$, $y = (1+0)^{\frac{1}{3}}$

$y = 1$

$$\text{Volume} = \int_1^2 \pi x^2 \, dy$$

$$y^3 = 1 + 2x^2$$

$$y^3 - 1 = 2x^2$$

$$\frac{y^3-1}{2} = x^2$$

$$\text{Volume} = \pi \int_1^2 \frac{y^3-1}{2} \, dy$$

$$= \frac{\pi}{2}\left[\frac{y^4}{4} - y\right]_1^2$$

$$= \frac{\pi}{2}\left[\left(\frac{2^4}{4} - 2\right) - \left(\frac{1^4}{4} - 1\right)\right]$$

$$= \frac{\pi}{2}\left[2 + \frac{3}{4}\right]$$

$$= \frac{11\pi}{8}$$

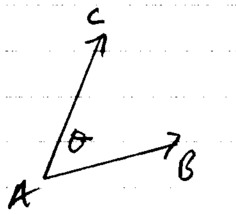
MEI CORE 4 JUNE 2014

7)

i) $A(-3, 0, 0)$
$B(2, 0, -2)$
$C(0, 4, 0)$
$D(0, 4, 5)$

$$|AB| = \sqrt{(2--3)^2 + 0^2 + (-2-0)^2} = \sqrt{29}$$

$$|AC| = \sqrt{(0--3)^2 + 4^2 + 0^2} = 5$$



$$\overrightarrow{AB} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix} \qquad \overrightarrow{AC} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$$

$$\cos\theta = \frac{\begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix} . \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}}{\left|\begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}\right| \left|\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}\right|} = \frac{15 + 0 + 0}{5\sqrt{29}}$$

$$\angle CAB = \theta = \cos^{-1}\left(\frac{3}{\sqrt{29}}\right) = 56.1^\circ$$

$$\text{Area} = \frac{1}{2} |AB||AC| \sin\theta$$

$$= \frac{1}{2} \times \sqrt{29} \times 5 \times \sin 56.1^\circ = 11.2 \text{ units}^2 \text{ to 3 s.f.}$$

ii)
A) $$\begin{pmatrix} 4 \\ -3 \\ 10 \end{pmatrix} . \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix} = 20 + 0 - 20 = 0$$

$\therefore (4\underline{i} - 3\underline{j} + 10\underline{k}) \perp$ to $\overrightarrow{AB}$

$$\begin{pmatrix} 4 \\ -3 \\ 10 \end{pmatrix} . \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} = 12 - 12 + 0 = 0$$

$\therefore (4\underline{i} - 3\underline{j} + 10\underline{k}) \perp$ to $\overrightarrow{AC}$

$(4\underline{i} - 3\underline{j} + 10\underline{k}) \perp$ to 2 non-parallel vectors in plane so normal to plane ABC

MEI CORE 4 JUNE 2014

7 ii) B) Plane of form $4x - 3y + 10z = d$

Sub $C(0,4,0)$ $0 - 12 + 0 = d$

$$-12 = d$$

Plane is $\underline{4x - 3y + 10z = -12}$

7 iii)

$$\underline{r} = \begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -3 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4\lambda \\ 4 - 3\lambda \\ 5 + 10\lambda \end{pmatrix}$$

Sub in plane

$$4(4\lambda) - 3(4 - 3\lambda) + 10(5 + 10\lambda) = -12$$

$$16\lambda - 12 + 9\lambda + 50 + 100\lambda = -12$$

$$125\lambda = -12 + 12 - 50 = -50$$

$$\lambda = -\frac{50}{125} = -0.4$$

Sub in line

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4(-0.4) \\ 4 - 3(-0.4) \\ 5 + 10(-0.4) \end{pmatrix} = \begin{pmatrix} -1.6 \\ 5.2 \\ 1 \end{pmatrix}$$

Point of intersection with ABC is $(-1.6, 5.2, 1)$

7iv) Volume $= \frac{1}{3} \times$ Area of $\triangle ABC \times$ height

$$= \frac{1}{3} \times 11.2 \times \perp \text{ distance of } D \text{ from plane}$$

$$= \frac{1}{3} \times 11.2 \times \sqrt{(0 - -1.6)^2 + (4 - 5.2)^2 + (5 - 1)^2}$$

$$= \frac{1}{3} \times 11.2 \times \sqrt{20}$$

$$= 16.7 \text{ units}^3 \text{ to } 3 \text{ s.f.}$$

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8) i) $h = \left(1 - \frac{1}{2}At\right)^2$

$$\Rightarrow \frac{dh}{dt} = 2\left(1 - \tfrac{1}{2}At\right) \times \left(-\tfrac{1}{2}A\right)$$

$$= -A\left(1 - \tfrac{1}{2}At\right)$$

$$= -A\sqrt{h}$$

Note $h = (1 - \frac{1}{2}At)^2$ also satisfies initial condition $h = 1,\ t = 0$

$\therefore h = (1 - \frac{1}{2}At)^2$ is the solution to the differential equation $\frac{dh}{dt} = -A\sqrt{h}$

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ii) $h = 0$ when $t = 20$

Substituting gives $0 = \left(1 - \frac{1}{2} \times A \times 20\right)^2$

$$\Rightarrow (1 - 10A) = 0$$

$$\Rightarrow A = 0.1$$

MEI CORE 4 JUNE 2014

8ii) cont) so $h = (1 - 0.05t)^2$

When $h = 0.5$

$$0.5 = (1 - 0.05t)^2$$

$$\sqrt{0.5} = 1 - 0.05t$$

$$0.05t = 1 - \sqrt{0.5}$$

$$t = \frac{1 - \sqrt{0.5}}{0.05}$$

$$t = 5.86\text{ s} \quad \text{to 3 s.f.}$$

8iii) $t = 0, h = 1$

$$\frac{dh}{dt} = -\frac{\beta\sqrt{h}}{(1+h)^2}$$

$$\int \frac{(1+h)^2}{\sqrt{h}}\,dh = \int -\beta\,dt$$

$$\int \frac{1 + 2h + h^2}{\sqrt{h}} = -\beta t + c$$

$$\int \left(h^{-\frac{1}{2}} + 2h^{\frac{1}{2}} + h^{3/2}\right) dh = -\beta t + c$$

$$\frac{h^{\frac{1}{2}}}{\frac{1}{2}} + \frac{2h^{3/2}}{3/2} + \frac{h^{5/2}}{5/2} = -\beta t + c$$

$$2h^{\frac{1}{2}} + \frac{4h^{3/2}}{3} + \frac{2h^{5/2}}{5} = -\beta t + c$$

MEI CORE 4 JUNE 2014

9iii) cont) When $t=0$, $h=1$

$$2 + \frac{4}{3} + \frac{2}{5} = c$$

$$\frac{30+20+6}{15} = c$$

$$\frac{56}{15} = c$$

$$\therefore \quad 2h^{\frac{1}{2}} + \frac{4h^{3/2}}{3} + \frac{2h^{5/2}}{5} = -Bt + \frac{56}{15}$$

$$\Rightarrow \quad 30h^{\frac{1}{2}} + 20h^{3/2} + 6h^{5/2} = -15Bt + 56$$

$$\Rightarrow \quad h^{\frac{1}{2}}(30 + 20h + 6h^2) = 56 - 15Bt$$

8iv) $h=0$, $t=20$

$$0 = 56 - 15B \times 20$$

$$0 = 56 - 300B$$

$$\Rightarrow B = \frac{56}{300} = \frac{14}{75}$$

Now $\quad h^{\frac{1}{2}}(30 + 20h + 6h^2) = 56 - 15 \times \frac{14}{75} t$

$h=0.5 \quad \sqrt{0.5}(30 + 20 \times 0.5 + 6 \times 0.5^2) = 56 - \frac{14}{5} t$

$$29.345 = 56 - 2.8t$$

$$2.8t = 56 - 29.345$$

$$t = \frac{56 - 29.345}{2.8} = 9.52 \text{ s} \quad \text{to } 3 \text{ s.f.}$$