MEI CORE 3 DIFFERENTIATION EXERCISE 4B

1) i) $y = (x^2-1)(x^3+3)$

$\frac{d}{dx} uv = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = (x^2-1)(3x^2) + (x^3+3)(2x)$

$= 3x^4 - 3x^2 + 2x^4 + 6x$

$= 5x^4 - 3x^2 + 6x$

ii) $y = x^5(3x^2+4x-7)$

$\frac{dy}{dx} = x^5(6x+4) + (3x^2+4x-7)5x^4$

$= 6x^6 + 4x^5 + 15x^6 + 20x^5 - 35x^4$

$= 21x^6 + 24x^5 - 35x^4$

iii) $y = x^2(2x+1)^4$

$\frac{dy}{dx} = x^2 \times 4(2x+1)^3 \times 2 + (2x+1)^4 \times 2x$

$= 8x^2(2x+1)^3 + 2x(2x+1)^4$

$= 8x^2(2x+1)^3 + 2x(2x+1)(2x+1)^3$

$= 8x^2(2x+1)^3 + (4x^2+2x)(2x+1)^3$

$= (12x^2+2x)(2x+1)^3$

$= 2x(6x+1)(2x+1)^3$

iv) $y = \frac{2x}{3x-1}$

$\frac{d}{dx} \frac{u}{v} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$\frac{dy}{dx} = \frac{(3x-1)\times 2 - 2x(3)}{(3x-1)^2}$

$\frac{dy}{dx} = \frac{6x-2-6x}{(3x-1)^2}$

$\frac{dy}{dx} = -\frac{2}{(3x-1)^2}$

v) $y = \frac{x^3}{x^2+1}$

$\frac{dy}{dx} = \frac{(x^2+1)\times 3x^2 - x^3 \times 2x}{(x^2+1)^2}$

$\frac{dy}{dx} = \frac{3x^4 + 3x^2 - 2x^4}{(x^2+1)^2}$

$\frac{dy}{dx} = \frac{x^4+3x^2}{(x^2+1)^2}$

$\frac{dy}{dx} = \frac{x^2(x^2+3)}{(x^2+1)^2}$

vi) $y = (2x+1)^2(3x^2-4)$

$\frac{dy}{dx} = (2x+1)^2(6x) + (3x^2-4)2(2x+1)2$

$= 6x(2x+1)^2 + 4(6x^3 - 8x + 3x^2 - 4)$

$= 6x(4x^2+4x+1) + 24x^3 - 32x + 12x^2 - 16$

$= 24x^3 + 24x^2 + 6x + 24x^3 - 32x + 12x^2 - 16$

$= 48x^3 + 36x^2 - 26x - 16$

vii) $y = \frac{2x-3}{2x^2+1}$

1 vii) cont

$$\frac{dy}{dx} = \frac{(2x^2+1)2 - (2x-3)4x}{(2x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{4x^2+2-8x^2+12x}{(2x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{12x-4x^2+2}{(2x^2+1)^2}$$

1 viii)

$$y = \frac{x-2}{(x+3)^2}$$

$$\frac{dy}{dx} = \frac{(x+3)^2 1 - (x-2)2(x+3)1}{(x+3)^4}$$

$$\frac{dy}{dx} = \frac{(x+3)^2 - (2x-4)(x+3)}{(x+3)^4}$$

$$\frac{dy}{dx} = \frac{(x+3)\big((x+3)-(2x-4)\big)}{(x+3)^4}$$

$$\frac{dy}{dx} = \frac{x+3-2x+4}{(x+3)^3}$$

$$\frac{dy}{dx} = \frac{7-x}{(x+3)^3}$$

1 ix)

$$y = (x+1)\sqrt{x-1}$$

$$y = (x+1)(x-1)^{\frac{1}{2}}$$

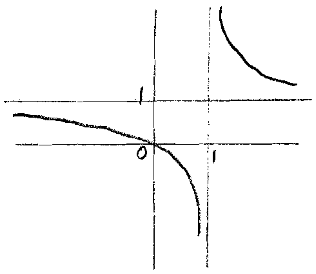
$$\frac{dy}{dx} = (x+1)\tfrac{1}{2}(x-1)^{-\frac{1}{2}}\times 1 + (x-1)^{\frac{1}{2}}\times 1$$

$$= \frac{x+1}{2\sqrt{x-1}} + \sqrt{x-1}$$

$$= \frac{(x+1)+2(x-1)}{2\sqrt{x-1}}$$

$$= \frac{3x-1}{2\sqrt{x-1}}$$

2) $y = \dfrac{x}{x-1}$



i)

$$\frac{dy}{dx} = \frac{(x-1)\times 1 - x(1)}{(x-1)^2}$$

$$\frac{dy}{dx} = \frac{x-1-x}{(x-1)^2}$$

$$\frac{dy}{dx} = -\frac{1}{(x-1)^2}$$

ii) when $x = 0$

gradient $= \dfrac{dy}{dx} = -\dfrac{1}{(-1)^2} = -1$

Using $y - y_1 = m(x - x_1)$

$$y - 0 = -1(x-0)$$

$$y = -x \quad \text{is tgt.}$$

iii) when $x = 2$

gradient $= \dfrac{dy}{dx} = -\dfrac{1}{(2-1)^2} = -1$

Using $y - y_1 = m(x - x_1)$

$$y - 2 = -1(x-2)$$

$$y - 2 = -x + 2$$

$$y = -x + 2 + 2$$

$$y = -x + 4 \quad \text{is tgt}$$

iv)

tgts are parallel

both have gradient $= -1$

3) $y = (x+1)(x-2)^2$

i) $\frac{dy}{dx} = (x+1)2(x-2) + (x-2)^2$

$= (2x+2)(x-2) + (x-2)(x-2)$

$= (2x+2+x-2)(x-2)$

$= 3x(x-2)$

ii) At st pt $\frac{dy}{dx} = 0$

$\Rightarrow 3x(x-2) = 0$

$\Rightarrow x = 0$ or $x = 2$

when $x = 0$, $y = 1(-2)^2 = 4$

when $x = 2$ $y = (3)0^2 = 0$

$\frac{dy}{dx} = 3x^2 - 6x$

$\therefore \frac{d^2y}{dx^2} = 6x - 6$

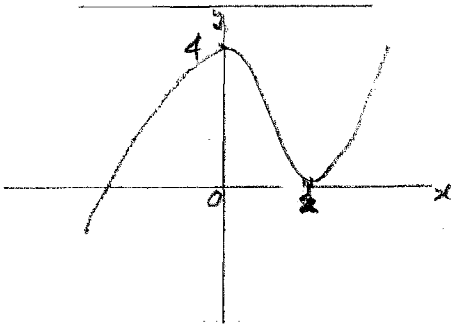
when $x = 0$, $\frac{d^2y}{dx^2} = -6$ a max

when $x = 2$, $\frac{d^2y}{dx^2} = +6$ a min

$\therefore$ a maximum occurs at $(0, 4)$

and a minimum occurs at $(2, 0)$

iii)



4) $y = \frac{x-3}{x-4}$

i) $\frac{dy}{dx} = \frac{(x-4)1 - (x-3)1}{(x-4)^2}$

$\frac{dy}{dx} = \frac{x-4-x+3}{(x-4)^2}$

$\frac{dy}{dx} = -\frac{1}{(x-4)^2}$

ii) When $x = 6$, $\frac{dy}{dx} = -\frac{1}{2^2} = -\frac{1}{4}$

Using $y - y_1 = m(x - x_1)$

$y - 1.5 = -\frac{1}{4}(x-6)$

$4y - 6 = -x + 6$

$x + 4y - 12 = 0$ is tgt

iii) When $x = 5$, $\frac{dy}{dx} = -\frac{1}{1^2} = -1$

$\therefore$ gradient of normal $= +1$

Using $y - y_1 = m(x - x_1)$

$y - 2 = 1(x-5)$

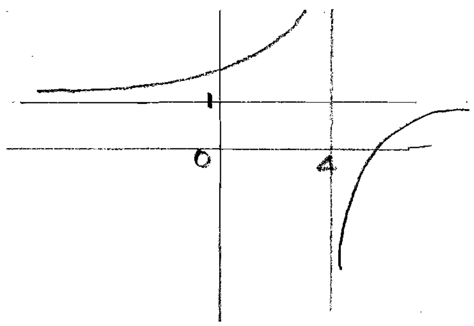
$y - 2 = x - 5$

$y = x - 3$ is normal at $(5, 2)$

iv) $-\frac{1}{(x-4)^2} \neq 0$ for all values of x

$\therefore$ no stationary points

4 iv) cont



when x just $< \frac{1}{2}$, $\frac{dy}{dx}$ is $+ \times + \times +$ +ve

when x just $> \frac{1}{2}$, $\frac{dy}{dx}$ is $+ \times + \times +$ +ve

Point of inflection when $x = \frac{1}{2}$

5) $y = (2x-1)^3(x+1)^3$

i) $\frac{dy}{dx} = (2x-1)^3 3(x+1)^2 + (x+1)^3 3(2x-1)^2 2$

$= 3(2x-1)^3(x+1)^2 + 6(x+1)^3(2x-1)^2$

$= 3(2x-1)^2(x+1)^2\left[(2x-1) + 2(x+1)\right]$

$= 3(2x-1)^2(x+1)^2(4x+1)$

ii) At st pt $\frac{dy}{dx} = 0$

$\Rightarrow 2x - 1 = 0 \qquad x = \frac{1}{2}$

or $(x+1) = 0 \qquad x = -1$

or $4x+1 = 0 \qquad x = -\frac{1}{4}$

When x just < -1, $\frac{dy}{dx}$ is $+ \times + \times -$ −ve

When x just > -1 $\frac{dy}{dx}$ is $+ \times + \times -$ −ve

Point of inflection when $x = -1$

when x just $< -\frac{1}{4}$, $\frac{dy}{dx}$ is $+ \times + \times -$ −ve

when x just $> -\frac{1}{4}$, $\frac{dy}{dx}$ is $+ \times + \times +$ +ve

A minimum when $x = -\frac{1}{4}$

iii)

$P = (-1, 0)$
$Q = \left(-\frac{1}{4}, -\frac{729}{512}\right)$
$R = \left(\frac{1}{2}, 0\right)$

when $x = -\frac{1}{4}$

$y = (-\frac{1}{2} - 1)^3\left(\frac{3}{4}\right)^3$

$y = \left(-\frac{3}{2}\right)^3\left(\frac{3}{4}\right)^3$

$= -\frac{27}{8} \times \frac{27}{64} = -\frac{729}{512}$

6) $y = \frac{2x}{\sqrt{x} - 1}$

i) $\frac{dy}{dx} = \frac{(\sqrt{x}-1)2 - 2x\frac{1}{2}x^{-\frac{1}{2}}}{(\sqrt{x}-1)^2}$

$\frac{dy}{dx} = \frac{2\sqrt{x} - 2 - \sqrt{x}}{(\sqrt{x}-1)^2}$

$\frac{dy}{dx} = \frac{\sqrt{x} - 2}{(\sqrt{x}-1)^2}$

ii) When $x = 9$

$\frac{dy}{dx} = \frac{3-2}{2^2} = \frac{1}{4}$

Gradient $= \frac{1}{4}$

6ii) cont Normal will have gradient -4

Using

$$y - y_1 = m(x - x_1)$$

$$y - 9 = -4(x - 9)$$

$$y - 9 = -4x + 36$$

$$y = -4x + 45$$

is normal

6 iii) At st.pt. $\frac{dy}{dx} = 0$

$$\Rightarrow \quad \frac{\sqrt{x} - 2}{(\sqrt{x} - 1)^2} = 0$$

$$\Rightarrow \quad \sqrt{x} - 2 = 0$$

$$\sqrt{x} = 2$$

$$x = 4$$

When $x = 4$ $\quad y = \frac{2 \times 4}{\sqrt{4} - 1} = \frac{8}{1}$

P is point $(4, 8)$

When x just < 4

$\frac{dy}{dx}$ is $\frac{-}{+} = -ve$

When x just > 4

$\frac{dy}{dx}$ is $\frac{+}{+} = +ve$

_/ $\therefore$ a minimum at $(4, 8)$

6iv) Tgt at P is $y = 8$

Normal at P is $x = 4$

6v)

$$\left.\begin{aligned} y &= -4x + 45 \\ y &= 8 \end{aligned}\right\}$$

$$8 = -4x + 45$$

$$4x = 37$$

$$x = \frac{37}{4}$$

Q is point $\left(\frac{37}{4}, 8\right)$

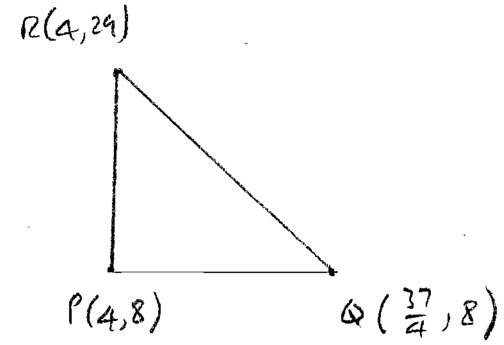
$$\left.\begin{aligned} y &= -4x + 45 \\ x &= 4 \end{aligned}\right\}$$

$$y = -16 + 45$$

$$y = 29$$

R is point $(4, 29)$

6vi)



Area $= \frac{1}{2}$ base $\times$ height

$$= \frac{1}{2} \times \frac{21}{4} \times 21$$

$$= \frac{441}{8}$$

7) $y = \dfrac{x^2 - 2x - 5}{2x + 3}$

i) $\dfrac{d}{dx}\dfrac{u}{v} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$\dfrac{dy}{dx} = \dfrac{(2x+3)(2x-2) - (x^2-2x-5)2}{(2x+3)^2}$

$= \dfrac{4x^2 + 6x - 4x - 6 - 2x^2 + 4x + 10}{(2x+3)^2}$

$= \dfrac{2x^2 + 6x + 4}{(2x+3)^2}$

$= \dfrac{2(x^2 + 3 + 2)}{(2x+3)^2}$

$= \dfrac{2(x+2)(x+1)}{(2x+3)^2}$

ii) At st. pt $\dfrac{dy}{dx} = 0$

$\Rightarrow 2(x+2)(x+1) = 0$

$\Rightarrow x = -2$ or $x = -1$

When $x = -2$ $y = \dfrac{(-2)^2 - 2(-2) - 5}{2(-2)+3}$

$y = \dfrac{4 + 4 - 5}{-4+3} = -3$

St. pt at $(-2, -3)$

when $x = -1$, $y = \dfrac{(-1)^2 - 2(-1) - 5}{2(-1)+3}$

$y = \dfrac{1 + 2 - 5}{1}$

$y = -2$

st. pt at $(-1, -2)$

iii) When x just < -2 $\dfrac{dy}{dx}$ is $\dfrac{- \times -}{+} = +ve$

When x just > -2 $\dfrac{dy}{dx}$ is $\dfrac{+ \times -}{+} = -ve$

$+ \diagup \overset{0}{-} \diagdown -$ $\therefore (-2, -3)$ a max

When x just < -1 $\dfrac{dy}{dx}$ is $\dfrac{+ \times -}{+} = -ve$

When x just > -1 $\dfrac{dy}{dx}$ is $\dfrac{+ \times +}{+} = +ve$

$- \diagdown \underset{0}{-} \diagup +$ $\therefore (-1, -2)$ a min

8) $y = \dfrac{x^2}{2x+1}$

$\dfrac{dy}{dx} = \dfrac{(2x+1)2x - x^2(2)}{(2x+1)^2}$

$= \dfrac{4x^2 + 2x - 2x^2}{(2x+1)^2}$

$= \dfrac{2x^2 + 2x}{(2x+1)^2}$

$= \dfrac{2x(x+1)}{(2x+1)^2}$

At st. pt $\dfrac{dy}{dx} = 0$

$\Rightarrow 2x(x+1) = 0$

$\Rightarrow x = 0$ or $x = -1$

8 cont) When $x = 0$ $y = \frac{0}{1} = 0$

when $x = -1$ $y = \frac{(-1)^2}{-2+1} = -1$

Stationary points at

$(0,0)$ and $(-1,-1)$

ii) Given that $\frac{d^2y}{dx^2} = \frac{2}{(2x+1)^3}$

when $x = 0$ $\frac{d^2y}{dx^2} = \frac{2}{1^3} = +2$

$\therefore (0,0)$ a min

when $x = -1$ $\frac{d^2y}{dx^2} = \frac{2}{(-1)^3} = -2$

$\therefore (-1,-1)$ a max

9) i) $x = y^2 + 4$

$\frac{dx}{dy} = 2y$

$\therefore \frac{dy}{dx} = \frac{1}{2y}$

ii)

$x = y^2 + 4$

$y^2 = x - 4$

$y = \pm\sqrt{x-4}$

$y = \pm(x-4)^{\frac{1}{2}}$

iii) $\frac{dy}{dx} = \pm\frac{1}{2}(x-4)^{-\frac{1}{2}} \times 1$

$\frac{dy}{dx} = \pm\frac{1}{2\sqrt{x-4}}$

This is the same as $\frac{1}{2y}$

iv)
Valid for $x \geq 4$

10) $f(x) = \frac{4x}{x^2+1}$

i) $f(0) = \frac{0}{1} = 0$

$f(1) = \frac{4}{2} = 2$

$f(2) = \frac{8}{5} = 1.6$

ii)

$f'(x) = \frac{(x^2+1)4 - 4x(2x)}{(x^2+1)^2}$

$= \frac{4x^2+4-8x^2}{(x^2+1)^2}$

$= \frac{4-4x^2}{(x^2+1)^2}$

$= \frac{4(1-x^2)}{(x^2+1)^2}$

$f'(x) = \frac{4(1+x)(1-x)}{(x^2+1)^2}$

At st. pt. $f'(x) = 0$

$\Rightarrow 4(1-x)(1+x) = 0$

$\Rightarrow x = 1$ or $x = -1$

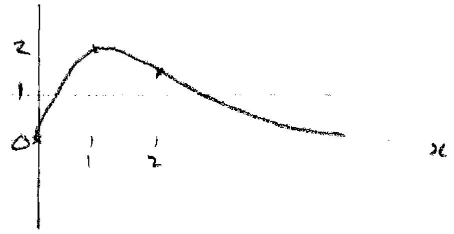
10 cont) $\therefore$ only one st pt when $x \geq 0$ which occurs when $x = 1$

when $x = 1$, $f(x) = 2$

$\therefore$ st. pt. at $(1, 2)$

iii) As $x \to \infty$, $f(x) \to 0$

iv)



v)

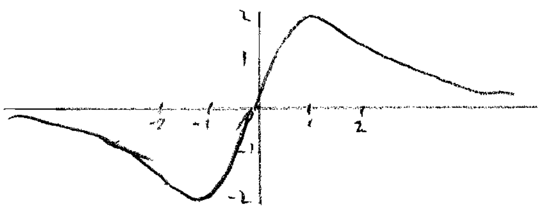
$$f(-x) = \frac{4(-x)}{(-x)^2 + 1}$$

$$= \frac{-4x}{x^2+1}$$

$$= -\frac{4x}{x^2+1}$$

$$= -f(x)$$

$\therefore$ $f(x)$ is an odd function



vi) Given
$g(x) = \frac{1}{x}$ $(x \neq 0)$

$$fg(x) = f\left(\tfrac{1}{x}\right)$$

$$= \frac{4\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)^2+1} = \frac{\frac{4}{x}}{\frac{1}{x^2}+1}$$

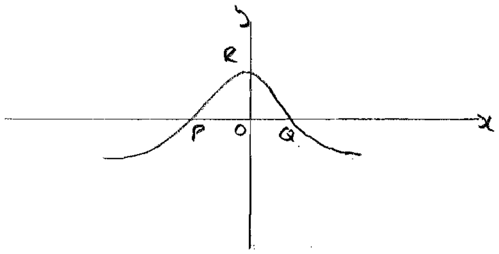
Mult by $\frac{x^2}{x^2}$

$$= \frac{\frac{4}{x} \times x^2}{\left(\frac{1}{x^2}+1\right) \times x^2} = \frac{4x}{1+x^2}$$

$$= f(x)$$

11)
$$f(x) = \frac{1-x^2}{1+x^2}$$

i)



$f(x) = 0 \Rightarrow 1 - x^2 = 0$
$\Rightarrow x = \pm 1$

$\therefore$ $P(-1, 0)$ $Q(1, 0)$

when $x = 0$ $f(x) = \frac{1-0}{1+0} = 1$

$\therefore$ $R(0, 1)$

As $x \to \infty$ $f(x) \to -1$

ii)
$$f(-x) = \frac{1-(-x)^2}{1+(-x)^2}$$

$$= \frac{1-x^2}{1+x^2} = f(x)$$

Symmetrical about y axis.

11 cont) iii) $f(x) = \dfrac{1-x^2}{1+x^2}$

$f'(x) = \dfrac{(1+x^2)\times -2x - (1-x^2)2x}{(1+x^2)^2}$

$f'(x) = \dfrac{-2x - 2x^3 - 2x + 2x^3}{(1+x^2)^2}$

$f'(x) = \dfrac{-4x}{(1+x^2)^2}$

$f'(-x) = \dfrac{-4(-x)}{(1+(-x)^2)^2}$

$= \dfrac{4x}{(1+x^2)^2}$

$= -\dfrac{-4x}{(1+x^2)^2}$

$= -f'(x)$

$\therefore f'(x)$ is an odd function

From symmetry of $f(x)$ we see that

$|f'(x)| = |f'(-x)|$

But $f'(x) > 0$ when $x < 0$
and $f'(x) < 0$ when $x > 0$

iv) $f''(x) = \dfrac{(1+x^2)^2 \times -4 + 4x \times 2(1+x^2)2x}{(1+x^2)^4}$

$f''(x) = \dfrac{-4(1+x^2)^2 + 16x^2(1+x^2)}{(1+x^2)^4}$

$f''(x) = \dfrac{-4(1+x^2) + 16x^2}{(1+x^2)^3}$

$f''(x) = \dfrac{12x^2 - 4}{(1+x^2)^3}$

$f''(x) = \dfrac{4(3x^2-1)}{(1+x^2)^3}$

$f''(x) = 0$ when $3x^2 - 1 = 0$

$\Rightarrow 3x^2 = 1$

$x^2 = \frac{1}{3}$

$x = \pm \frac{1}{\sqrt{3}}$

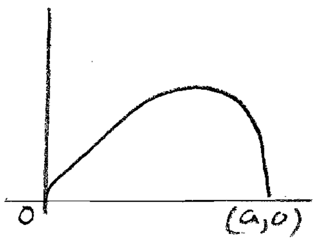
When $x = \pm\frac{1}{\sqrt{3}}$ $f(x) = \dfrac{1-\frac{1}{3}}{1+\frac{1}{3}}$

$f(x) = \dfrac{\frac{2}{3}}{\frac{4}{3}} = \frac{2}{3} \times \frac{3}{4} = \frac{1}{2}$

$f''(x) = 0$ at $\left(\frac{1}{\sqrt{3}}, \frac{1}{2}\right)$ and $\left(-\frac{1}{\sqrt{3}}, \frac{1}{2}\right)$

These are points of inflection

12) $y = x\sqrt{9 - 2x^2}$



12 cont)

i) when $y=0$, $x\sqrt{9-2x^2}=0$

$$\Rightarrow x=0 \text{ or } 9-2x^2=0$$

$$\Rightarrow 2x^2=9$$

$$x^2=\frac{9}{2}$$

$$x=\frac{3}{\sqrt{2}}$$

$$x=\frac{3\sqrt{2}}{2}$$

$$\therefore a=\frac{3\sqrt{2}}{2}$$

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ii)

$$\frac{d}{dx}\sqrt{9-2x^2}=\frac{d}{dx}(9-2x^2)^{\frac{1}{2}}$$

$$=\frac{1}{2}(9-2x^2)^{-\frac{1}{2}}\times -4x$$

$$=\frac{-2x}{\sqrt{9-2x^2}}$$

$$y=x\sqrt{9-2x^2}$$

$$\frac{dy}{dx}=x\times\frac{-2x}{\sqrt{9-2x^2}}+\sqrt{9-2x^2}$$

$$=\frac{-2x^2}{\sqrt{9-2x^2}}+\frac{(9-2x^2)}{\sqrt{9-2x^2}}$$

$$=\frac{9-4x^2}{\sqrt{9-2x^2}}$$

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iii) At max point $\frac{dy}{dx}=0$

$$\Rightarrow 9-4x^2=0$$

$$\Rightarrow 4x^2=9$$

$$x^2=\frac{9}{4}$$

At max pt $x=\frac{3}{2}$

When $x=0$, $\frac{dy}{dx}=\frac{9}{\sqrt{9}}=\frac{9}{3}=3$

At origin gradient $=3$

At $(a,0)=\left(\frac{3\sqrt{2}}{2},0\right)$

$$\frac{dy}{dx}=\frac{9-4(\frac{9}{2})}{\sqrt{9-2(\frac{9}{2})}}$$

$$=\frac{-9}{\sqrt{0}}$$

$\therefore$ gradient is infinite
