11) a) i)

$$\int_1^6 \frac{1}{2x+3}\,dx$$

$$= \frac{1}{2}\int_1^6 \frac{2}{2x+3}\,dx$$

$$= \frac{1}{2}\Big[\ln(2x+3)\Big]_1^6$$

$$= \frac{1}{2}\Big[\ln 15 - \ln 5\Big]$$

$$= \frac{1}{2}\ln\frac{15}{5} = \frac{1}{2}\ln 3$$

b)

$$\int \frac{x}{\sqrt{9+x^2}}\,dx$$

Let $u = 9+x^2$

$\Rightarrow \frac{du}{dx} = 2x$

$\Rightarrow du = 2x\,dx$

$\Rightarrow \frac{1}{2}du = x\,dx$

$$\int \frac{x}{\sqrt{9+x^2}}\,dx = \int \frac{1}{2}\frac{1}{u^{\frac{1}{2}}}\,du$$

$$= \int \frac{1}{2}u^{-\frac{1}{2}}\,du$$

$$= \frac{\frac{1}{2}u^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= u^{\frac{1}{2}} + c$$

$$= \sqrt{9+x^2} + c$$

b) Using the recommended substitution $v = \sqrt{9+x^2}$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2}(9+x^2)^{-\frac{1}{2}} \times 2x$$

$$\frac{dv}{dx} = \frac{x}{\sqrt{9+x^2}}$$

$$dv = \frac{x}{\sqrt{9+x^2}}\,dx$$

$$\therefore \int \frac{x}{\sqrt{9+x^2}}\,dx = \int 1\,dv$$

$$= v + c$$

$$= \sqrt{9+x^2} + c$$

11 ii)

a) $\frac{d}{dx}e^{-x^2}$

Let $u = -x^2$

$\frac{du}{dx} = -2x$

Note that $\frac{d}{dx} = \frac{d}{du}\frac{du}{dx}$

Find $\frac{d}{dx}e^u = \frac{d}{du}e^u \times \frac{du}{dx}$

$$= e^u \times -2x$$

$$= -2xe^{-x^2}$$

b)

$$y = xe^{-x^2}$$

$$\frac{dy}{dx} = x \times -2xe^{-x^2} + e^{-x^2} \times 1$$

$$= e^{-x^2}(1-2x^2)$$

At st pts $\frac{dy}{dx} = 0$

$$\Rightarrow 1-2x^2 = 0$$

$$1 = 2x^2$$

$$\frac{1}{2} = x^2$$

$$x = \pm\frac{1}{\sqrt{2}}$$

11 cont) When $x = \frac{1}{\sqrt{2}}$ $y = \frac{1}{\sqrt{2}} e^{-\frac{1}{2}}$

st pt $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2e}}\right)$

When $x = \frac{-1}{\sqrt{2}}$, $y = \frac{-1}{\sqrt{2}} e^{-\frac{1}{2}}$

st. pt $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2e}}\right)$

c) Area $= \int_0^{0.4} x e^{-x^2}\,dx$

Let $u = x^2$

$\Rightarrow \frac{du}{dx} = 2x$

$\Rightarrow du = 2x\,dx$

$\Rightarrow \frac{1}{2} du = x\,dx$

When $x = 0.4$, $u = 0.16$

when $x = 0$, $u = 0$

$\int_0^{0.4} x e^{-x^2}\,dx = \int_0^{0.16} \frac{1}{2} e^{-u}\,du$

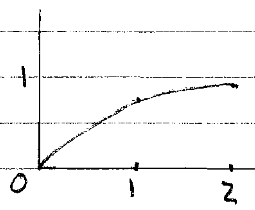
$= \left[-\frac{1}{2} e^{-u}\right]_0^{0.16}$

$= -\frac{1}{2} e^{-0.16} + \frac{1}{2} e^0$

$= 0.0739$ to 3 sig fig

12

i) $y = \frac{e^x}{e^x+1}$



ii) Area $= \int_0^2 \frac{e^x}{e^x+1}\,dx$

Let $u = e^x + 1$

$\Rightarrow \frac{du}{dx} = e^x$

$\Rightarrow du = e^x\,dx$

When $x = 2$, $u = e^2+1$

when $x = 0$, $u = 2$

$\int_0^2 \frac{e^x}{e^x+1}\,dx = \int_2^{e^2+1} \frac{1}{u}\,du$

$= \left[\ln u\right]_2^{e^2+1}$

$= \ln(e^2+1) - \ln 2$

$= \ln\left(\frac{e^2+1}{2}\right)$

$= 1.43$ to 3 sig fig

iii) $\int_0^e \frac{2t}{t^2+1}\,dt$

$= \left[\ln(t^2+1)\right]_0^e$

$= \ln(e^2+1) - \ln 1$

$= \ln(e^2+1)$

$= 2.13$ to 3 sig fig

12 iv) Answer in book claims answers to parts (ii) and (iii) are the same because substitution $e^x = t^2$ transforms part (ii) to part (iii).

This is incorrect as part (iii) would need to be $\int_1^e \frac{2t}{t^2+1}\,dt$

instead of $\int_0^e \frac{2t}{t^2+1}\,dt$ as it is in the question

$\therefore$ an error in book

13 i)

$$\int_5^{10} \frac{1}{u}\,du = \Big[\ln u\Big]_5^{10}$$

$$= \ln 10 - \ln 5$$

$$= \ln\left(\frac{10}{5}\right) = \ln 2$$

13 ii)

$$f(x) = \frac{x}{x^2+1}$$

$$\int_2^3 \frac{x}{x^2+1}\,dx$$

$$= \frac{1}{2}\int_2^3 \frac{2x}{x^2+1}\,dx$$

$$= \frac{1}{2}\Big[\ln(x^2+1)\Big]_2^3$$

$$= \frac{1}{2}\left(\ln 10 - \ln 5\right)$$

$$= \frac{1}{2}\ln\left(\frac{10}{5}\right) = \frac{1}{2}\ln 2$$

13 iii)

$$f(-x) = \frac{(-x)}{(-x)^2+1}$$

$$= \frac{(-x)}{x^2+1}$$

$$= -\frac{x}{x^2+1} = -f(x)$$

$\therefore f(x)$ is an odd function

$$\int_{-3}^{-2} f(x)\,dx$$

$$= -\frac{1}{2}\ln 2$$

Since $f(x)$ is an odd function.

13 iv) $y = f(x)$ transformed into $y = f(x+1)$ by the translation $\begin{pmatrix}-1\\0\end{pmatrix}$

13 v)

$$\int_{-4}^{-3} \frac{x+1}{x^2+2x+2}\,dx$$

$$= \int_{-4}^{-3} \frac{x+1}{(x+1)^2+1}\,dx$$

Let $u = x+1$

$\frac{du}{dx} = 1$

$du = dx$

when $x = -3$, $u = -2$

when $x = -4$, $u = -3$

13 v) cont)

$$\int_{-4}^{-3} \frac{x+1}{(x+1)^2+1}\,dx = \int_{-3}^{-2} \frac{u}{u^2+1}\,du$$

$$= -\tfrac{1}{2}\ln 2$$

from part (iii)

14)

i) a) $\dfrac{d}{dx} e^{-2x^2} = -4x e^{-2x^2}$

b) $\dfrac{d}{dx} x e^{-2x^2} = x \times (-4x e^{-2x^2}) + e^{-2x^2} \times 1$

$$= (1-4x^2)e^{-2x^2}$$

14) ii)

$$f(x) = x e^{-2x^2}$$

$$\int_0^k f(x)\,dx$$

$$= \int_0^k x e^{-2x^2}\,dx$$

Let $u = 2x^2$

$\Rightarrow \dfrac{du}{dx} = 4x$

$\Rightarrow du = 4x\,dx$

$\Rightarrow \frac{1}{4}\,du = x\,dx$

when $x = k$, $u = 2k^2$

when $x = 0$, $u = 0$

$$\int_0^k x e^{-2x^2}\,dx = \int_0^{2k^2} \frac{1}{4} e^{-u}\,du$$

$$= \frac{1}{4}\Big[-e^{-u}\Big]_0^{2k^2}$$

$$= \frac{1}{4}\Big[-e^{-2k^2} + e^0\Big]$$

$$= \frac{1}{4}\left(1 - e^{-2k^2}\right)$$

14 iii)

From i) b)

$$f'(x) = (1-4x^2)e^{-2x^2}$$

$$f''(x) = (1-4x^2) \times -4x e^{-2x^2} + e^{-2x^2} \times (-8x)$$

$$= (-4x + 16x^3 - 8x)e^{-2x^2}$$

$$= (-12x + 16x^3)e^{-2x^2}$$

$$= 4x(4x^2-3)e^{-2x^2}$$

For st. pt $f'(x) = 0$

$\Rightarrow 1 - 4x^2 = 0$

$\Rightarrow 4x^2 = 1$

$x^2 = \frac{1}{4}$

$x = \pm\frac{1}{2}$

Only one solution for $x > 0$

$x = \frac{1}{2}$, $y = \frac{1}{2}e^{-\frac{1}{2}}$

$\left(\frac{1}{2}, \frac{1}{2\sqrt{e}}\right)$

When $x = \frac{1}{2}$,

$f''(\tfrac{1}{2}) = 4 \times \tfrac{1}{2}(4 \times \tfrac{1}{4} - 3)e^{-\frac{1}{2}}$

$= 2(-2)e^{-\frac{1}{2}} = -4e^{-\frac{1}{2}}$

< 0 $\therefore$ st pt is a max.

15 i) Area OABE $= \int_0^{\ln 5} e^x \, dx$

$= \left[e^x \right]_0^{\ln 5}$

$= e^{\ln 5} - e^0$

$= 5 - 1 = 4$ units^2

Area of rectangle OABC

$=$ length $\times$ width $= 5 \ln 5$

Shaded area $= 5 \ln 5 - 4$

15 ii)

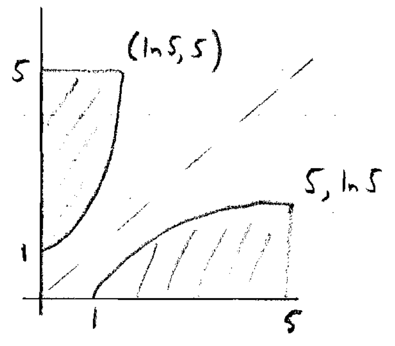
a) $y = e^x$ transformed into $y = \ln x$ by a reflection in line $y = x$

b) If $y = \ln(x^3)$

$y = 3 \ln x$

Obtained from $y = \ln x$ by a stretch of scale factor 3 parallel to y-axis

15 iii)



$\int_1^5 \ln x \, dx = 5 \ln 5 - 4$

same as area found on graph of inverse fn $y = e^x$

(0, 1) corresponding point to (1, 0)

(5, ln5) corresponding point to (ln5, 5)

iv) $\int_1^5 \ln(x^3) \, dx$

$= \int_1^5 3 \ln x \, dx = 3 \int_1^5 \ln x \, dx$

$= 15 \ln 5 - 12$

$\int_1^5 \ln(3x) \, dx$

$= \int_1^5 (\ln x + \ln 3) \, dx$

$= \int_1^5 \ln x \, dx + \int_1^5 \ln 3 \, dx$

$= 5 \ln 5 - 4 + \left[x \ln 3 \right]_1^5$

$= 5 \ln 5 - 4 + 4 \ln 3$

16) $y = 5 e^{-0.08x}$

i) when $x = 20$

$y = 5 \times e^{-0.08 \times 20}$

$= 5 \times e^{-1.6}$

$= 1.009$

Height of Q $= 1.009 \times 100$ m

$= 101$ m (n'est m)

16 ii)

$$y = 5e^{-0.08x}$$

$$\frac{dy}{dx} = -0.08 \times 5e^{-0.08x}$$

$$= -0.4e^{-0.08x}$$

Gradient at P where $x = 0$

$$= -0.4e^0$$

$$= -0.4$$

(and since vert and horizontal scales the same)

16 iii)

$$\text{Area} = \int_0^{20} 5e^{-0.08x}\,dx$$

$$= \left[\frac{5e^{-0.08x}}{-0.08}\right]_0^{20}$$

$$= \left[-62.5e^{-0.08x}\right]_0^{20}$$

$$= -62.5e^{-1.6} + 62.5e^0$$

$$= 49.88 \text{ units}^2$$

16 iv)

$$\int_0^{20} \frac{25}{5+x}\,dx$$

$$= 25\left[\ln(x+5)\right]_0^{20}$$

$$= 25(\ln 25 - \ln 5)$$

$$= 25\ln\left(\frac{25}{5}\right) = 25\ln 5$$

$$= 40.24 \text{ units}^2$$

Area removed

$$\approx 49.88 - 40.24$$

$$= 9.64 \text{ units}^2$$

Real area of cross-section

$$9.64 \times 100 \times 100 \text{ m}^2$$

Volume of rock removed

$$= 9.64 \times 100 \times 100 \times 5$$

$$= 482{,}000 \text{ m}^3$$

$$= 480{,}000 \text{ m}^3 \text{ to 2 sig fig}$$

17)

i)

$$y = \frac{x}{x^2+1}$$

$$\frac{dy}{dx} = \frac{(x^2+1)\times 1 - x(2x)}{(x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{x^2+1-2x^2}{(x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{1-x^2}{(x^2+1)^2}$$

ii) At st. pt. $\frac{dy}{dx} = 0$

$$\Rightarrow 1 - x^2 = 0$$

$$\Rightarrow x = \pm 1$$

When $x = 1$, $y = \frac{1}{1^2+1} = \frac{1}{2}$

When $x = -1$ $y = \frac{-1}{(-1)^2+1} = -\frac{1}{2}$

Turning points $\left(1, \frac{1}{2}\right)$ and $\left(-1, -\frac{1}{2}\right)$

17 cont) iii)

$$\int \frac{x}{x^2+1}\,dx$$

$$= \frac{1}{2}\int \frac{2x}{x^2+1}\,dx$$

$$= \frac{1}{2}\ln|x^2+1| + c$$

17 iv)

Let P have x-coord X

then $\int_0^X \frac{x}{x^2+1}\,dx = 10$

$$\left[\frac{1}{2}\ln|x^2+1|\right]_0^X = 10$$

$$\frac{1}{2}\left[\ln(X^2+1) - \ln 1\right] = 10$$

$$\ln(X^2+1) - 0 = 20$$

$$X^2+1 = e^{20}$$

$$X^2 = e^{20} - 1$$

$$X = \sqrt{e^{20}-1}$$

$$X = 22,026$$

to best whole number

18)

$$y = f(x) = \frac{e^x}{1+e^x}$$

i) At P $x = 0$

$$y = \frac{e^0}{1+e^0} = \frac{1}{1+1} = \frac{1}{2}$$

P is point $\left(0, \frac{1}{2}\right)$

As $x \to \infty$ $\frac{e^x}{1+e^x} \to 1$

ℓ is line $y = 1$

18 ii)

$$f(x) = \frac{e^x}{1+e^x}$$

$$f'(x) = \frac{(1+e^x)e^x - e^x(e^x)}{(1+e^x)^2}$$

$$f'(x) = \frac{e^x + e^{2x} - e^{2x}}{(1+e^x)^2}$$

$$f'(x) = \frac{e^x}{(1+e^x)^2}$$

Gradient at P

$$= \frac{e^0}{(1+e^0)^2} = \frac{1}{2^2} = \frac{1}{4}$$

18 iii)

$$\text{Area} = \int_0^1 \frac{e^x}{1+e^x}\,dx$$

$$= \left[\ln(1+e^x)\right]_0^1$$

$$= \ln(1+e) - \ln(1+1)$$

$$= \ln(1+e) - \ln 2$$

$$= \ln\left(\frac{1+e}{2}\right)$$

18 iv)

$$f(x) + f(-x)$$

$$= \frac{e^x}{1+e^x} + \frac{e^{-x}}{1+e^{-x}}$$

18 cont iv)

$$\frac{e^x(1+e^{-x})+e^{-x}(1+e^x)}{(1+e^x)(1+e^{-x})}$$

$$= \frac{e^x+1+e^{-x}+1}{1+e^x+e^{-x}+1}$$

$$= 1$$

Graph has rotational symmetry of order 2 about $P\ (0,\tfrac{1}{2})$

19) i) $f(x) = \dfrac{x}{1+x^2}$

$$f(-x) = \frac{(-x)}{1+(-x)^2}$$

$$= \frac{(-x)}{1+x^2}$$

$$= -\frac{x}{1+x^2}$$

$$= -f(x)$$

$\therefore f(x)$ is an odd function.

Rotational symmetry of order 2 about $(0,0)$

19ii)

$$y = \frac{x}{1+x^2}$$

$$\frac{dy}{dx} = \frac{(1+x^2)\times 1 - x(2x)}{(1+x^2)^2}$$

$$\frac{dy}{dx} = \frac{1+x^2-2x^2}{(1+x^2)^2}$$

$$\frac{dy}{dx} = \frac{1-x^2}{(1+x^2)^2}$$

At st. pt $\dfrac{dy}{dx} = 0$

$$\Rightarrow 1-x^2 = 0$$

$$\Rightarrow x = \pm 1$$

When $x = 1$, $y = \dfrac{1}{1+1^2} = \dfrac{1}{2}$

When $x = -1$, $y = \dfrac{-1}{1+(-1)^2} = \dfrac{-1}{2} = -\dfrac{1}{2}$

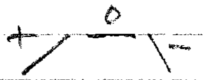
St pts at $(1, \tfrac{1}{2})$ and $(-1, -\tfrac{1}{2})$

When x is just < 1 (say 0.99)

$$\frac{dy}{dx} = \frac{1-0.99^2}{(1+0.99^2)^2} > 0$$

When x is just > 1 (say 1.1)

$$\frac{dy}{dx} = \frac{1-(1.1)^2}{(1+1.1^2)^2} < 0$$



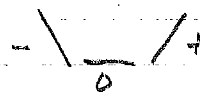
$\therefore (1, \tfrac{1}{2})$ is a max

When x is just < -1 (say -1.1)

$$\frac{dy}{dx} = \frac{1-(-1.1)^2}{(1+(-1.1)^2)^2} < 0$$

When x is just > -1 (say -0.9)

$$\frac{dy}{dx} = \frac{1-(-0.9)^2}{(1+(-0.9)^2)^2} > 0$$



$\therefore (-1, -\tfrac{1}{2})$ is a min

19 iii)

$$\text{Area} = \int_0^1 f(x)\,dx$$

19 cont) iii)

$$\int_0^1 \frac{x}{1+x^2}\,dx$$

$$= \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx$$

$$= \frac{1}{2}\Big[\ln|1+x^2|\Big]_0^1$$

$$= \frac{1}{2}\Big[\ln 2 - \ln 1\Big]$$

$$= \frac{1}{2}\ln 2$$

20)

$$y = \frac{x}{(2x-1)^2}$$

i) $a = \frac{1}{2}$

(since when $x = \frac{1}{2}$, denominator becomes 0)

ii)

$$\frac{dy}{dx} = \frac{(2x-1)^2 \times 1 - x \times 2(2x-1) \times 2}{(2x-1)^4}$$

$$\frac{dy}{dx} = \frac{(2x-1)(2x-1) - 4x(2x-1)}{(2x-1)^4}$$

$$\frac{dy}{dx} = \frac{(2x-1)(2x-1-4x)}{(2x-1)^4}$$

$$\frac{dy}{dx} = \frac{-1-2x}{(2x-1)^3}$$

$$\frac{dy}{dx} = -\frac{(2x+1)}{(2x-1)^3}$$

At st. pt. P $\frac{dy}{dx} = 0$

$$\Rightarrow 2x+1 = 0$$

$$\Rightarrow 2x = -1$$

$$x = -\frac{1}{2}$$

When $x = -\frac{1}{2}$

$$y = \frac{-\frac{1}{2}}{(-1-1)^2} = \frac{-\frac{1}{2}}{4} = -\frac{1}{8}$$

P is point $\left(-\frac{1}{2}, -\frac{1}{8}\right)$

20 iii)

$$A = \int_1^2 \frac{x}{(2x-1)^2}\,dx$$

Let $u = 2x-1$

$$\Rightarrow \frac{du}{dx} = 2$$

$$\Rightarrow du = 2dx$$

$$\frac{1}{2}du = dx$$

Also $u+1 = 2x$

$$\frac{u+1}{2} = x$$

When $x = 2$, $u = 2\times 2 - 1 = 3$

When $x = 1$, $u = 2\times 1 - 1 = 1$

$$\int_1^2 \frac{x}{(2x-1)^2}\,dx = \int_1^3 \frac{(u+1)}{2}\frac{1}{u^2}\frac{1}{2}\,du$$

$$= \int_1^3 \frac{u+1}{4u^2}\,du$$

$$= \frac{1}{4}\int_1^3 \frac{u+1}{u^2}\,du = \frac{1}{4}\int_1^3\left(\frac{1}{u}+\frac{1}{u^2}\right)du$$

$$= \frac{1}{4}\left[\ln u - \frac{1}{u}\right]_1^3$$

$$= \frac{1}{4}\left[\left(\ln 3 - \frac{1}{3}\right) - (\ln 1 - 1)\right]$$

$$= \frac{1}{4}\left[\ln 3 + \frac{2}{3}\right] = \frac{3\ln 3 + 2}{12}$$