1) i) $\int \frac{3}{x}\,dx = 3\ln|x| + c$

ii) $\int \frac{1}{4x}\,dx = \frac{1}{4}\ln|x| + c$

iii) $\int \frac{1}{x-5}\,dx = \ln|x-5| + c$

iv) $\int \frac{1}{2x-9}\,dx = \frac{1}{2}\int \frac{2}{2x-9}\,dx$

$= \frac{1}{2}\ln|2x-9| + c$

v) $\int \frac{2x}{x^2+1}\,dx = \ln|x^2+1| + c$

vi) $\int \frac{2x+3}{3x^2+9x-1}\,dx$

$= \frac{1}{3}\int \frac{6x+9}{3x^2+9x-1}\,dx$

$= \frac{1}{3}\ln|3x^2+9x-1| + c$

2) i) $\int e^{3x}\,dx = \frac{1}{3}e^{3x} + c$

ii) $\int e^{-4x}\,dx = -\frac{1}{4}e^{-4x} + c$

iii) $\int e^{-\frac{x}{3}}\,dx = -3e^{-\frac{x}{3}} + c$

iv) $\int 12x^2 e^{x^3}\,dx$

Let $u = x^3$

$\Rightarrow \frac{du}{dx} = 3x^2$

$\Rightarrow du = 3x^2\,dx$

$\int 12x^2 e^{x^3}\,dx$

$= \int 4e^u\,du$

$= 4e^u + c$

$= 4e^{x^3} + c$

v) $\int \frac{10}{e^{5x}}\,dx = \int 10e^{-5x}\,dx$

$= \frac{10e^{-5x}}{-5} + c$

$= -2e^{-5x} + c$

vi) $\int \frac{e^{3x}+4}{e^{2x}}\,dx$

$= \int\left(\frac{e^{3x}}{e^{2x}} + \frac{4}{e^{2x}}\right)dx$

$= \int (e^x + 4e^{-2x})\,dx$

$= e^x + \frac{4e^{-2x}}{-2} + c$

$= e^x - 2e^{-2x} + c$

3) i) $\int_0^4 4e^{2x}\,dx$

$= \left[\frac{4e^{2x}}{2}\right]_0^4$

$= \left[2e^{2x}\right]_0^4 = 2e^8 - 2$

MEI CORE 3 INTEGRATION INVOLVING e^x $\ln x$ EXERCISE 5B

3ii) $\displaystyle\int_1^3 \frac{4}{2x+1}\,dx = 2\int_1^3 \frac{2}{2x+1}\,dx$

$= 2\left[\ln|2x+1|\right]_1^3$

$= 2\left[\ln 7 - \ln 3\right]$

$= 2\ln\left(\frac{7}{3}\right)$

3iii) $\displaystyle\int_2^3 2xe^{-x^2}\,dx$

Let $u = x^2$

$\Rightarrow \frac{du}{dx} = 2x$

$\Rightarrow du = 2x\,dx$

When $x = 3$, $u = 9$
when $x = 2$, $u = 4$

$\displaystyle\int_2^3 2xe^{-x^2}\,dx = \int_4^9 e^{-u}\,du$

$\left[-e^{-u}\right]_4^9$

$= \left(-e^{-9}\right) - \left(-e^{-4}\right)$

$= \frac{1}{e^4} - \frac{1}{e^9} = 0.018$ to 3 sig fig

3iv) $\displaystyle\int_{-1}^1 (e^x + e^{-x})\,dx$

$= \left[e^x - e^{-x}\right]_{-1}^1$

$= \left(e - \frac{1}{e}\right) - \left(\frac{1}{e} - e\right)$

$= 2e - \frac{2}{e} = 4.70$ to 3 sig fig

3v) $\displaystyle\int_{-2}^1 e^{3x-2}\,dx$

Let $u = 3x - 2$

$\Rightarrow \frac{du}{dx} = 3$

$du = 3dx$

$\frac{1}{3}du = dx$

When $x = 1$, $u = 3\times1 - 2 = 1$
when $x = -2$, $u = 3(-2) - 2 = -8$

$\displaystyle\int_{-2}^1 e^{3x-2}\,dx = \int_{-8}^1 \frac{1}{3}e^u\,du$

$= \frac{1}{3}\left[e^u\right]_{-8}^1$

$= \frac{1}{3}\left[e - e^{-8}\right]$

$= 0.906$ to 3 sig fig

3vi) $\displaystyle\int_2^4 \frac{x+2}{x^2+4x-3}\,dx$

$= \frac{1}{2}\displaystyle\int_2^4 \frac{2x+4}{x^2+4x-3}\,dx$

$= \frac{1}{2}\left[\ln|x^2+4x-3|\right]_2^4$

$= \frac{1}{2}\left[\ln 29 - \ln 9\right]$

$= 0.585$ to 3 sig fig

MEI CORE 3 INTEGRATION INVOLVING e^x, $\ln x$ EXERCISE 5B

4) i) $y = xe^{x^2}$

Area $A = \left| \int_{-1}^{0} xe^{x^2}\,dx \right|$

$\int_{-1}^{0} xe^{x^2}\,dx$

Let $u = x^2$

$\Rightarrow \frac{du}{dx} = 2x$

$\Rightarrow du = 2x\,dx$

$\Rightarrow \frac{1}{2}du = x\,dx$

When $x = 0$, $u = 0$

when $x = -1$, $u = 1$

$$\int_{-1}^{0} xe^{x^2}\,dx = \int_{1}^{0} \frac{1}{2}e^u\,du$$

$$= \left[\frac{1}{2}e^u\right]_1^0$$

$$= \frac{1}{2}(1-e)$$

Area $A = \left|\frac{1}{2}(1-e)\right|$

$$= \frac{1}{2}(e-1)$$

4 ii) Area $B = \int_0^2 xe^{x^2}\,dx$

Same substitution

when $x = 2$ $u = 4$

when $x = 0$ $u = 0$

$$\int_0^2 xe^{x^2}\,dx = \int_0^4 \frac{1}{2}e^u\,du$$

$$= \left[\frac{1}{2}e^u\right]_0^4$$

$$= \frac{1}{2}(e^4-1)$$

4 iii)

Total Area of shaded region

$$= \frac{1}{2}(e-1) + \frac{1}{2}(e^4-1)$$

$$= \frac{1}{2}(e^4+e-2)$$

5) $y = xe^{-x^2}$

i) $\frac{dy}{dx} = x(-2xe^{-x^2}) + e^{-x^2}\times 1$

$$= -2x^2e^{-x^2} + e^{-x^2}$$

$$= (1-2x^2)e^{-x^2}$$

ii) At st. pt $\frac{dy}{dx} = 0$

$\Rightarrow (1-2x^2) = 0$

$1 = 2x^2$

$\frac{1}{2} = x^2$

$x = \pm\frac{1}{\sqrt{2}}$

$x = \pm\frac{\sqrt{2}}{2}$

Max at $x = +\frac{\sqrt{2}}{2}$

5 cont) iii) $\text{Area} = \int_{\frac{1}{\sqrt{2}}}^{2} x e^{-x^2}\,dx$

Let $u = x^2$

$\Rightarrow \frac{du}{dx} = 2x$

$\Rightarrow du = 2x\,dx$

$\Rightarrow \frac{1}{2}du = x\,dx$

when $x = 2$, $u = 4$

when $x = \frac{1}{\sqrt{2}}$, $u = \frac{1}{2}$

$$\int_{\frac{1}{\sqrt{2}}}^{2} x e^{-x^2}\,dx = \int_{\frac{1}{2}}^{4} \frac{1}{2}e^{-u}\,du$$

$$= \left[-\frac{1}{2}e^{-u}\right]_{\frac{1}{2}}^{4}$$

$$= \frac{1}{2}\left[-e^{-4} - \left(-e^{-\frac{1}{2}}\right)\right]$$

$$= \frac{1}{2}\left[e^{-\frac{1}{2}} - e^{-4}\right]$$

$= 0.294$ to 3 sig fig.

6) $$y = \frac{x+2}{x^2+4x+3}$$

First shaded region

$$= \left| \int_{-5}^{-4} \frac{x+2}{x^2+4x+3}\,dx \right|$$

$$= \left| \frac{1}{2}\int_{-5}^{-4} \frac{2x+4}{x^2+4x+3}\,dx \right|$$

$$= \left| \frac{1}{2}\left[\ln|x^2+4x+3|\right]_{-5}^{-4} \right|$$

$$= \left| \frac{1}{2}\left[\ln 3 - \ln 8\right] \right|$$

$$= \left| -0.4904 \right|$$

$= 0.490$ to 3 sig fig

Second shaded region

$$= \int_{1}^{2} \frac{x+2}{x^2+4x+3}\,dx$$

$$= \frac{1}{2}\int_{1}^{2} \frac{2x+4}{x^2+4x+3}\,dx$$

$$= \frac{1}{2}\left[\ln|x^2+4x+3|\right]_{1}^{2}$$

$$= \frac{1}{2}\left(\ln 15 - \ln 8\right)$$

$= 0.314$ to 3 sig fig

7) $$y = x + \frac{4}{x}$$

i) $$\frac{dy}{dx} = 1 - \frac{4}{x^2}$$

$$\frac{dy}{dx} = \frac{x^2-4}{x^2}$$

At st. pt $\frac{dy}{dx} = 0$

7 cont) i)

$\Rightarrow x^2 - 4 = 0$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$

From graph min pt P when $x=2$

max point Q when $x=-2$

When $x = 2$, $y = 2 + \frac{4}{2} = 4$

$\therefore$ P is point $(2,4)$

When $x = -2$, $y = -2 + \frac{4}{-2}$

$= -4$

$\therefore$ Q is point $(-2,-4)$

7ii)

Area of first shaded region

$$= \left| \int_{-4}^{-2} \left(x + \frac{4}{x}\right) dx \right|$$

$$= \left| \left[\frac{x^2}{2} + 4\ln|x| \right]_{-4}^{-2} \right|$$

$$= \left| (2 + 4\ln 2) - (8 + 4\ln 4) \right|$$

$$= \left| -6 + 4\ln 2 - 4\ln 4 \right|$$

$$= \left| -8.773 \right|$$

$= 8.77$ to 3 sig fig

Area of second shaded region

$$= \int_2^5 \left(x + \frac{4}{x}\right) dx$$

$$= \left[\frac{x^2}{2} + 4\ln x \right]_2^5$$

$$= \left(\frac{25}{2} + 4\ln 5\right) - \left(2 + 4\ln 2\right)$$

$$= \frac{21}{2} + 4\ln 5 - 4\ln 2$$

$$= 14.165$$

$= 14.2$ to 3 sig fig

8) i)

$$\int_0^X x e^{-x^2} dx$$

Let $u = x^2$

$\Rightarrow \frac{du}{dx} = 2x$

$\Rightarrow du = 2x\,dx$

$\Rightarrow \frac{1}{2} du = x\,dx$

When $x = X$, $u = X^2$

when $x = 0$, $u = 0$

$$\int_0^X x e^{-x^2} dx = \int_0^{X^2} \frac{1}{2} e^{-u} du$$

$$= \left[-\frac{1}{2} e^{-u} \right]_0^{X^2}$$

$$= -\frac{1}{2} e^{-X^2} - \left(-\frac{1}{2} e^0\right)$$

8 cont) i) $= -\frac{1}{2}e^{-x^2} + \frac{1}{2}$

$= \frac{1}{2}\left(1 - e^{-x^2}\right)$

8ii) For $X = 1$ Integral $= 0.3161$

$X = 2$ Integral $= 0.4908$

$X = 3$ Integral $= 0.4999$

$X = 4$ Integral $= 0.5000$

iii) As $X \to \infty$ $e^{-x^2} \to 0$

$\frac{1}{2}\left(1 - e^{-x^2}\right) \to \frac{1}{2}(1 - 0)$

Integral $\to \frac{1}{2}$

9) $y = (x+3)e^{-x}$

i) $\frac{dy}{dx} = (x+3)(-e^{-x}) + e^{-x} \times 1$

$= (-x - 3 + 1)e^{-x}$

$= (-x - 2)e^{-x}$

$\frac{dy}{dx} = -(x+2)e^{-x}$

9 ii) At st pt $\frac{dy}{dx} = 0$

$\Rightarrow x + 2 = 0$

$\Rightarrow x = -2$

When $x = -2$

$y = (-2+3)e^2 = e^2$

Stationary point S

is $(-2, e^2)$

9 iii) $\frac{dy}{dx} = -(x+2)e^{-x}$

$= (-x-2)e^{-x}$

$\frac{d^2y}{dx^2} = (-x-2)(-e^{-x}) + e^{-x}(-1)$

$= (x+2)(e^{-x}) - e^{-x}$

$= (x+1)e^{-x}$

At S when $x = -2$

$\frac{d^2y}{dx^2} = (-2+1)e^2$

$= -e^2$

Indicates S is a maximum

9 iv) $\int \frac{2 + \ln u}{u^2}\, du$

Let $e^x = u$

$\Rightarrow x = \ln u$

$\Rightarrow \frac{dx}{du} = \frac{1}{u}$

$\Rightarrow dx = \frac{du}{u}$

$\therefore \int \frac{2 + \ln u}{u^2}\, du = \int \frac{2 + x}{e^x}\, dx$

when $u = e$, $x = 1$

when $u = 1$, $x = 0$

$\therefore \int_1^e \frac{2 + \ln u}{u^2}\, du = \int_0^1 \frac{2 + x}{e^x}\, dx$

9(iv) cont)

$$= \int_0^1 \left((2+x)e^{-x}\right) dx$$

Let $u = 2+x$ Let $\frac{dv}{dx} = e^{-x}$

$\Rightarrow \frac{du}{dx} = 1$ $\Rightarrow v = -e^{-x}$

Using $\int u \frac{dv}{dx} = uv - \int v \frac{du}{dx}$

$$\int_0^1 (2+x)e^{-x} dx$$

$$= \left[(2+x)(-e^{-x})\right]_0^1 - \int_0^1 -e^{-x} dx$$

$$= \left[(2+x)(-e^{-x})\right]_0^1 - \left[e^{-x}\right]_0^1$$

$$= \left[-3e^{-1} + 2\right] - \left[e^{-1} - 1\right]$$

$$= -4e^{-1} + 3$$

$$= 3 - \frac{4}{e}$$

10) i)

$$\int 2x\sqrt{2x-3}\, dx$$

Let $u^2 = 2x-3$

$2u\frac{du}{dx} = 2$

Also $2x = u^2 + 3$

$2u\,du = 2dx$

$u\,du = dx$

$$\int 2x\sqrt{2x-3}\, dx = \int (u^2+3)\, u\, u\, du$$

$$= \int (u^4 + 3u^2)\, du$$

$$= \frac{u^5}{5} + u^3 + c$$

$$= \frac{(2x-3)^{5/2}}{5} + (2x-3)^{3/2} + c$$

An alternative and possibly easier substitution would be let $u = 2x - 3$

$$\int 2x\sqrt{2x-3}\, dx$$

Let $u = 2x-3$

$\frac{du}{dx} = 2$

$du = 2dx$

$\frac{1}{2}du = dx$

Also since $u = 2x-3$

$u+3 = 2x$

$$\int 2x\sqrt{2x-3}\, dx = \int (u+3)\, u^{\frac{1}{2}} \frac{1}{2}\, du$$

$$= \int \left(\frac{1}{2}u^{3/2} + \frac{3}{2}u^{\frac{1}{2}}\right) du$$

$$= \frac{1}{2}\frac{u^{5/2}}{\frac{5}{2}} + \frac{3}{2}\frac{u^{3/2}}{\frac{3}{2}} + c$$

$$= \frac{2}{5}\cdot\frac{1}{2}u^{5/2} + \frac{2}{3}\cdot\frac{3}{2}u^{3/2} + c$$

$$= \frac{u^{5/2}}{5} + u^{3/2} + c$$

$$= \frac{(2x-3)^{5/2}}{5} + (2x-3)^{3/2} + c$$

Same result of course!

10 cont) ii) $\frac{d}{dx} x^{\frac{1}{2}} \ln x$

$= x^{\frac{1}{2}} \frac{1}{x} + \ln x \times \frac{1}{2} x^{-\frac{1}{2}}$

$= x^{-\frac{1}{2}} + \frac{1}{2} \ln x \times x^{-\frac{1}{2}}$

$= x^{-\frac{1}{2}} \left(1 + \frac{1}{2} \ln x\right)$

$= \frac{1}{2} x^{-\frac{1}{2}} \left(2 + \ln x\right)$

$= \frac{2 + \ln x}{2\sqrt{x}}$

$\int \frac{2 + \ln x}{\sqrt{x}} dx$

$= 2 \int \frac{2 + \ln x}{2\sqrt{x}} dx$

$= 2 x^{\frac{1}{2}} \ln x + c$

(from first part)

10iii) $f'(x) = e^{-x^2}$

a) $f''(x) = -2x e^{-x^2}$

b) $\frac{d}{dx} f(x^3)$

$= 3x^2 f'(x^3)$

$= 3x^2 e^{-x^6}$

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