MEI CORE 3 INTEGRATION BY PARTS EXERCISE 5D

1) i) $\int x e^x \, dx$

Let $u = x$ $\Rightarrow \frac{du}{dx} = 1$

Let $\frac{dv}{dx} = e^x$ $\Rightarrow v = e^x$

Using $\int u \frac{dv}{dx} = uv - \int v \frac{du}{dx}$

$\int x e^x \, dx = x e^x - \int 1 e^x \, dx$

$= x e^x - e^x + c$

1ii) $\int x \cos 3x \, dx$

Let $u = x$ $\Rightarrow \frac{du}{dx} = 1$

Let $\frac{dv}{dx} = \cos 3x$ $\Rightarrow v = \frac{1}{3} \sin 3x$

Using

$\int u \frac{dv}{dx} = uv - \int v \frac{du}{dx}$

$\int x \cos 3x \, dx = \frac{1}{3} x \sin 3x - \int \frac{1}{3} \sin 3x \, dx$

$= \frac{1}{3} x \sin 3x + \frac{1}{9} \cos 3x + c$

1iii) $\int (2x+1) \cos x \, dx$

Let $u = 2x+1$ $\Rightarrow \frac{du}{dx} = 2$

Let $\frac{dv}{dx} = \cos x$ $\Rightarrow v = \sin x$

Using

$\int u \frac{dv}{dx} = uv - \int v \frac{du}{dx}$

$\int (2x+1) \cos x \, dx$

$= (2x+1) \sin x - \int 2 \sin x \, dx$

$= (2x+1) \sin x + 2 \cos x + c$

1iv) $\int x e^{-2x} \, dx$

Let $u = x$ $\Rightarrow \frac{du}{dx} = 1$

Let $\frac{dv}{dx} = e^{-2x}$ $\Rightarrow v = -\frac{1}{2} e^{-2x}$

Using $\int u \frac{dv}{dx} = uv - \int v \frac{du}{dx}$

$\int x e^{-2x} \, dx$

$= -\frac{1}{2} x e^{-2x} + \int \frac{1}{2} e^{-2x} \, dx$

$= -\frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x} + c$

1v) $\int x e^{-x} \, dx$

Let $u = x$ $\Rightarrow \frac{du}{dx} = 1$

Let $\frac{dv}{dx} = e^{-x}$ $\Rightarrow v = -e^{-x}$

Using $\int u \frac{dv}{dx} = uv - \int v \frac{du}{dx}$

$\int x e^{-x} \, dx = -x e^{-x} + \int e^{-x} \, dx$

$= -x e^{-x} - e^{-x} + c$

1vi) $\int x \sin 2x \, dx$

1vi) (cont)

Let $u = x$ $\Rightarrow \frac{du}{dx} = 1$

Let $\frac{dv}{dx} = \sin 2x$ $\Rightarrow v = -\frac{1}{2}\cos 2x$

Using $\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$

$$\int x\sin 2x\,dx = -\frac{1}{2}x\cos 2x + \int \frac{1}{2}\cos 2x\,dx$$

$$= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + c$$

2i)

$$\int x^3 \ln x\,dx$$

Let $u = \ln x$ $\Rightarrow \frac{du}{dx} = \frac{1}{x}$

Let $\frac{dv}{dx} = x^3$ $\Rightarrow v = \frac{x^4}{4}$

Using $\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$

$$\int x^3 \ln x\,dx = \frac{1}{4}x^4\ln x - \int \frac{x^4}{4}\cdot\frac{1}{x}\,dx$$

$$= \frac{1}{4}x^4\ln x - \int \frac{x^3}{4}\,dx$$

$$= \frac{1}{4}x^4\ln x - \frac{x^4}{16} + c$$

2ii)

$$\int 3xe^{3x}\,dx$$

Let $u = 3x$ $\Rightarrow \frac{du}{dx} = 3$

Let $\frac{dv}{dx} = e^{3x}$ $\Rightarrow v = \frac{1}{3}e^{3x}$

Using $\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$

$$\int 3xe^{3x}\,dx = xe^{3x} - \int e^{3x}\,dx$$

$$= xe^{3x} - \frac{1}{3}e^{3x} + c$$

2iii)

$$\int 2x\cos 2x\,dx$$

Let $u = 2x$ $\Rightarrow \frac{du}{dx} = 2$

Let $\frac{dv}{dx} = \cos 2x$ $\Rightarrow v = \frac{1}{2}\sin 2x$

Using $\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$

$$\int 2x\cos 2x\,dx = x\sin 2x - \int \sin 2x\,dx$$

$$= x\sin 2x + \frac{1}{2}\cos 2x + c$$

2iv)

$$\int x^2\ln 2x\,dx$$

Let $u = \ln 2x$ $\Rightarrow \frac{du}{dx} = \frac{2}{2x}$, $\frac{du}{dx} = \frac{1}{x}$

Let $\frac{dv}{dx} = x^2$ $\Rightarrow v = \frac{x^3}{3}$

Using $\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$

$$\int x^2\ln 2x\,dx = \frac{1}{3}x^3\ln 2x - \int \frac{x^3}{3}\frac{1}{x}\,dx$$

2 iv) (cont) $= \frac{1}{3}x^3 \ln 2x - \int \frac{x^2}{3} dx$

$= \frac{1}{3}x^3 \ln 2x - \frac{x^3}{9} + c$

3 $\int x\sqrt{1+x}\, dx$ by parts

$= \int x(1+x)^{\frac{1}{2}} dx$

Let $u = x$ Let $\frac{dv}{dx} = (1+x)^{\frac{1}{2}}$

$\Rightarrow \frac{du}{dx} = 1$ $\Rightarrow v = \frac{(1+x)^{3/2}}{\frac{3}{2}}$

$v = \frac{2}{3}(1+x)^{3/2}$

Using $\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$

$\int x\sqrt{1+x}\, dx$

$= \frac{2}{3}x(1+x)^{3/2} - \int \frac{2}{3}(1+x)^{3/2} dx$

$= \frac{2}{3}x(1+x)^{3/2} - \frac{2}{3}\frac{(1+x)^{5/2}}{\frac{5}{2}} + c$

$= \frac{2}{3}x(1+x)^{3/2} - \frac{4}{15}(1+x)^{5/2} + c$

3) $\int x\sqrt{1+x}$ by substitution

Let $u = 1+x$

$\Rightarrow \frac{du}{dx} = 1$

$du = dx$

Also $x = u - 1$

$\int x\sqrt{1+x}\, dx$

$= \int (u-1)u^{\frac{1}{2}} du$

$= \int (u^{3/2} - u^{\frac{1}{2}}) du$

$= \frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{\frac{3}{2}} + c$

$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + c$

$= \frac{2}{5}(1+x)^{5/2} - \frac{2}{3}(1+x)^{3/2} + c$

We can show these answers are equivalent

$\frac{2}{5}(1+x)^{5/2} - \frac{2}{3}(1+x)^{3/2} + c$

$= \frac{6}{15}(1+x)(1+x)^{3/2} - \frac{10}{15}(1+x)^{3/2} + c$

$= \frac{1}{15}(1+x)^{3/2}[6(1+x) - 10] + c$

$= \frac{1}{15}(1+x)^{3/2}[6 + 6x - 10] + c$

$= \frac{1}{15}(1+x)^{3/2}(6x - 4) + c$

$= \frac{2}{15}(1+x)^{3/2}(3x - 2) + c$

3 cont) Also integration by parts answer

$$\frac{2}{3}x(1+x)^{3/2} - \frac{4}{15}(1+x)^{5/2} + c$$

$$= \frac{10}{15}x(1+x)^{3/2} - \frac{4}{15}(1+x)(1+x)^{3/2} + c$$

$$= \frac{1}{15}(10x - 4 - 4x)(1+x)^{\frac{3}{2}} + c$$

$$= \frac{1}{15}(6 - 4x)(1+x)^{3/2} + c$$

$$= \frac{2}{15}(3 - 2x)(1+x)^{3/2} + c$$

4) i) $\int 2x(x-2)^4\,dx$

Let $u = 2x$ Let $\frac{dv}{dx} = (x-2)^4$

$\Rightarrow \frac{du}{dx} = 2$

$\Rightarrow v = \frac{(x-2)^5}{5}$

Using $\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$

$$\int 2x(x-2)^4\,dx$$

$$= \frac{2}{5}x(x-2)^5 - \int \frac{2(x-2)^5}{5}$$

$$= \frac{2}{5}x(x-2)^5 - \frac{2}{30}(x-2)^6 + c$$

$$= \frac{6}{15}x(x-2)^5 - \frac{1}{15}(x-2)(x-2)^5 + c$$

$$= \frac{1}{15}(x-2)^5(6x - x + 2) + c$$

$$= \frac{1}{15}(x-2)^5(5x+2) + c$$

ii) $\int 2x(x-2)^4\,dx$

Let $u = x - 2$

$\Rightarrow \frac{du}{dx} = 1$

$\Rightarrow du = dx$

Also $u + 2 = x$

$$\int 2x(x-2)^4\,dx$$

$$= \int 2(u+2)u^4\,du$$

$$= \int (2u^5 + 4u^4)\,du$$

$$= \frac{2u^6}{6} + \frac{4u^5}{5} + c$$

$$= \frac{1}{3}(x-2)^6 + \frac{4}{5}(x-2)^5 + c$$

$$= \frac{5}{15}(x-2)(x-2)^5 + \frac{12}{15}(x-2)^5 + c$$

$$= \frac{1}{15}(x-2)^5\left[5x - 10 + 12\right] + c$$

$$= \frac{1}{15}(x-2)^5(5x+2) + c$$

5) i) $\int \ln x\,dx$

$$= \int 1 \times \ln x\,dx$$

Let $u = \ln x$ Let $\frac{dv}{dx} = 1$

$\Rightarrow \frac{du}{dx} = \frac{1}{x}$ $\Rightarrow v = x$

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5i) (cont) Using $\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$

$\int \ln x\,dx = x\ln x - \int x.\frac{1}{x}\,dx$

$= x\ln x - \int 1\,dx$

$= x\ln x - x + c$

5ii)

$\int \ln 3x\,dx = \int 1 \times \ln 3x\,dx$

Let $u = \ln 3x$ Let $\frac{dv}{dx} = 1$

$\Rightarrow \frac{du}{dx} = \frac{3}{3x}$ $\Rightarrow v = x$

$\frac{du}{dx} = \frac{1}{x}$

Using $\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$

$\int \ln 3x\,dx = x\ln 3x - \int \frac{1}{x}x\,dx$

$= x\ln 3x - \int 1\,dx$

$= x\ln 3x - x + c$

5iii)

$\int \ln px\,dx \quad p > 0$

$= x\ln px - x + c$

ENRICHMENT MATERIAL

NOT ON EXAM

(QUESTIONS 6 AND 7)

6) $\int x^2 e^x\,dx$

(Need to use integration by parts twice to illiminate the x^2 term from the integral)

Let $u = x^2$ Let $\frac{dv}{dx} = e^x$

$\Rightarrow \frac{du}{dx} = 2x$ $\Rightarrow v = e^x$

Using $\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$

$\int x^2 e^x\,dx = x^2e^x - \int 2xe^x\,dx$ (1)

Now consider $\int 2xe^x\,dx$

Let $u = 2x$ Let $\frac{dv}{dx} = e^x$

$\Rightarrow \frac{du}{dx} = 2$ $\Rightarrow v = e^x$

Using $\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$

$\int 2xe^x\,dx = 2xe^x - \int 2e^x\,dx$

$= 2xe^x - 2e^x + c$

Substituting this answer in (1) gives

$\int x^2e^x\,dx$

$= x^2e^x - 2xe^x + 2e^x + c$

(No need to write $-c$ as c is any constant)

7) $\int (2-x)^2 \cos x \, dx$

Need to use integration by parts twice

Let $u = (2-x)^2$ Let $\frac{dv}{dx} = \cos x$

$\Rightarrow \frac{du}{dx} = -2(2-x)$ $\Rightarrow v = \sin x$

Using $\int u \frac{dv}{dx} = uv - \int v \frac{du}{dx}$

$\int (2-x)^2 \cos x \, dx = (2-x)^2 \sin x + \int 2(2-x) \sin x \, dx$ (1)

Now consider

$\int 2(2-x) \sin x \, dx$

$= \int (4-2x) \sin x \, dx$

Let $u = 4-2x$ Let $\frac{dv}{dx} = \sin x$

$\Rightarrow \frac{du}{dx} = -2$ $\Rightarrow v = -\cos x$

Using $\int u \frac{dv}{dx} = uv - \int v \frac{du}{dx}$

$\int (4-2x) \sin x \, dx$

$= -\cos x(4-2x) - \int (-2)(-\cos x) dx$

$= -(4-2x)\cos x - \int 2\cos x \, dx$

$= -(4-2x)\cos x - 2\sin x + c$

Subst this answer in (1)

$\int (2-x)^2 \cos x \, dx$

$= (2-x)^2 \sin x - (4-2x)\cos x - 2\sin x + c$

$= (2-x)^2 \sin x - 2(2-x)\cos x - 2\sin x + c$