MEI FP1 MATRICES EXERCISE 1D

1) i) $\underline{Y} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ reflection in x axis

Rotation $\theta°$ anticlockwise $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

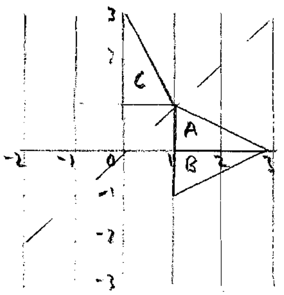
$\underline{Q} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ rotation $90°$ anticlockwise

ii) $\underline{Q}\,\underline{Y} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Reflection in line $y = x$

iii)



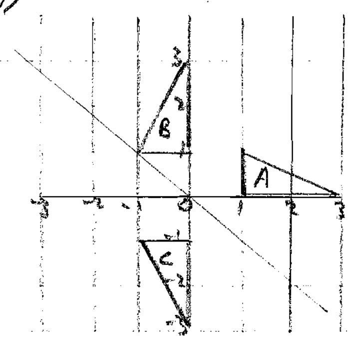
A reflected in x axis gives B
B rotated $90°$ anticlockwise about $(0,0)$ gives C
A reflected in $y=x$ also gives C

iv) $\underline{Y}\,\underline{Q} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Reflection in line $y = -x$

v)



A rotated $90°$ anticlockwise about $(0,0)$ gives B. B reflected in x axis gives C. A reflected in line $y=-x$ also gives C.

2)
i) Rotation by $\theta°$ clockwise about $(0,0)$ is given by $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

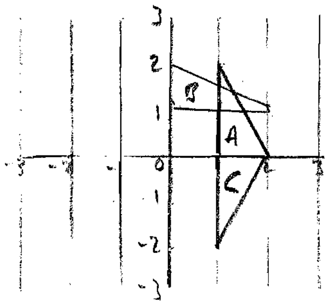
$\underline{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ rotation $90°$ clockwise about $(0,0)$

$\underline{T} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ reflection in line $y = x$

ii) $\underline{R}\,\underline{T} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Reflection in x axis

iii)



A reflected in line $y=x$ gives B
B rotated $90°$ clockwise about $(0,0)$ gives C
A reflected in x axis also gives C

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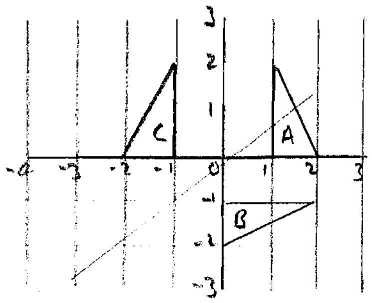
iv) $\underline{T}\underline{R} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Reflection in y axis

v)



A rotated 90° clockwise about $(0,0)$ gives B. B reflected in line $y=x$ gives C
A reflected in y axis also gives C

3)

i) $\underline{X} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ reflection in x axis

$\underline{Y} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ reflection in y axis

(ii) $\underline{X}\,\underline{Y} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Rotation by 180° about $(0,0)$

$\underline{Y}\,\underline{X} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

iv) Same since both represent a rotation by 180° about $(0,0)$

4)

i) $\underline{S} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ rotation by 180° about $(0,0)$

$\underline{U} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ reflection in line $y=-x$

ii) $\underline{S}\,\underline{U} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Reflection in line $y=x$

iii) $\underline{U}\,\underline{S} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

iv) Both these transformations in either order result in a reflection in the line $y=x$

5)

$\underline{R} = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$ $\underline{S} = \begin{pmatrix} 3 & 0 \\ -2 & 4 \end{pmatrix}$

i) $\underline{R}\,\underline{S} = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 8 & -4 \\ -3 & 12 \end{pmatrix}$

ii) $\underline{R}\,\underline{S}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 & -4 \\ -3 & 12 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 20 \\ -18 \end{pmatrix}$

Image $= (20, -18)$

6) $\underline{R}_1$ rotate $25°$ anticlockwise about $(0,0)$

i)

$\underline{R}_2$ rotate $40°$ anticlockwise about $(0,0)$

Both $\underline{R}_1\underline{R}_2$ and $\underline{R}_2\underline{R}_1$ are equivalent to a rotation of $65°$ anticlockwise about $(0,0)$ and are therefore equal

ii) Rotation by $\theta°$ anticlockwise about $(0,0)$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\underline{R}_1 = \begin{pmatrix} \cos 25° & -\sin 25° \\ \sin 25° & \cos 25° \end{pmatrix}$$

$$\underline{R}_2 = \begin{pmatrix} \cos 40° & -\sin 40° \\ \sin 40° & \cos 40° \end{pmatrix}$$

$\underline{R}_1\underline{R}_2 =$

$$\begin{pmatrix} \cos 25 \cos 40 - \sin 25 \sin 40 & -\cos 25 \sin 40 - \sin 25 \cos 40 \\ 0.4226 & -0.9063 \\ \\ \sin 25° \cos 40° + \cos 25 \sin 40° & -\sin 25° \sin 40° + \cos 25° \cos 40° \\ 0.9063 & 0.4226 \end{pmatrix}$$

iii) Rotation by $65°$ anticlockwise about $(0,0)$

7)

i)

$$V_2 = V_1 - I_1 R_1$$

$$I_2 = I_1$$

$$\begin{pmatrix} V_2 \\ I_2 \end{pmatrix} = \underline{A} \begin{pmatrix} V_1 \\ I_1 \end{pmatrix}$$

$$\underline{A} = \begin{pmatrix} 1 & -R_1 \\ 0 & 1 \end{pmatrix}$$

ii)

$$V_2 = V_1$$

$$I_2 = I_1 - \frac{V_1}{R_2}$$

$$\begin{pmatrix} V_2 \\ I_2 \end{pmatrix} = \underline{B} \begin{pmatrix} V_1 \\ I_1 \end{pmatrix}$$

$$\underline{B} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{R_2} & 1 \end{pmatrix}$$

iii)

$$\underline{B}\,\underline{A} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{R_2} & 1 \end{pmatrix}\begin{pmatrix} 1 & -R_1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -R_1 \\ -\frac{1}{R_2} & \frac{R_1}{R_2}+1 \end{pmatrix}$$

iv)

$$\underline{A}\,\underline{B} = \begin{pmatrix} 1 & -R_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{R_2} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+\frac{R_1}{R_2} & -R_1 \\ -\frac{1}{R_2} & 1 \end{pmatrix}$$

$$\underline{A}\underline{B} \neq \underline{B}\underline{A}$$

$\therefore$ effects not the same.

8) $\frac{1}{1+m^2}\begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}$

i) $\underline{P}$ is reflection in $y = \frac{1}{\sqrt{3}}x$

$$\underline{P} = \frac{1}{1+\frac{1}{3}}\begin{pmatrix} 1-\frac{1}{3} & \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} & \frac{1}{3}-1 \end{pmatrix}$$

$$= \frac{1}{4/3}\begin{pmatrix} \frac{2}{3} & \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} & -\frac{2}{3} \end{pmatrix}$$

$$= \frac{3}{4}\begin{pmatrix} \frac{2}{3} & \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} & -\frac{2}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

Q is reflection in $y = \sqrt{3}x$

$$\underline{Q} = \frac{1}{1+3}\begin{pmatrix} 1-3 & 2\sqrt{3} \\ 2\sqrt{3} & 3-1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} -2 & 2\sqrt{3} \\ 2\sqrt{3} & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

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ii) $\underline{Q}\,\underline{P} =$

$$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$\cos 60 = \frac{1}{2}$ $\quad = \begin{pmatrix} \cos 60 & -\sin 60 \\ \sin 60 & \cos 60 \end{pmatrix}$

Rotation by $60°$ anti-clockwise about $(0,0)$

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iii) $\underline{P}\,\underline{Q} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

$$= \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

Rotation by $60°$ clockwise about $(0,0)$

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9)

i) $\underline{R} = \begin{pmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{pmatrix}$

$$= \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

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ii) From 8(i)

$$M = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

9 iii) $\underline{MR} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Reflection in line $y = x$

10) $\underline{T} = \frac{1}{1+m^2} \begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}$

$\underline{T}^2 = \frac{1}{(1+m^2)^2} \begin{pmatrix} (1-m^2)^2 + 4m^2 & (1-m^2)2m + 2m(m^2-1) \\ (1-m^2)2m + (m^2-1)2m & 4m^2 + (m^2-1)^2 \end{pmatrix}$

$= \frac{1}{(1+m^2)^2} \begin{pmatrix} 1-2m^2+m^4+4m^2 & 0 \\ 0 & 4m^2+m^4-2m^2+1 \end{pmatrix}$

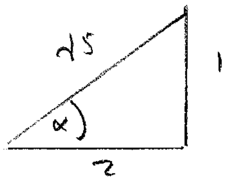
$= \frac{1}{(1+m^2)^2} \begin{pmatrix} 1+2m^2+m^4 & 0 \\ 0 & 1+2m^2+m^4 \end{pmatrix}$

$= \frac{1}{(1+m^2)^2} \begin{pmatrix} (1+m^2)^2 & 0 \\ 0 & (1+m^2)^2 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$

Reflecting in same line twice causes shape to revert to original position

11) i) $\tan\alpha = \frac{1}{2}$



$\cos\alpha = \frac{2}{\sqrt{5}}$ $\sin\alpha = \frac{1}{\sqrt{5}}$

A = rotate $\alpha°$ clockwise about $(0,0)$

$\underline{A} = \begin{pmatrix} \cos\alpha° & \sin\alpha° \\ -\sin\alpha° & \cos\alpha° \end{pmatrix}$

$\underline{A} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$

B = one way stretch ×5 parallel to x axis

$\underline{B} = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$

C = rotate $\alpha°$ anti-clockwise about $(0,0)$

$\underline{C} = \begin{pmatrix} \cos\alpha° & -\sin\alpha° \\ \sin\alpha° & \cos\alpha° \end{pmatrix}$

$\underline{C} = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$

11 cont)

ii) $\underline{S} = \underline{C}\underline{B}\underline{A}$

$$\underline{C}\underline{B} = \begin{pmatrix} \frac{2}{\sqrt5} & -\frac{1}{\sqrt5} \\ \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2\sqrt5 & -\frac{1}{\sqrt5} \\ \sqrt5 & \frac{2}{\sqrt5} \end{pmatrix}$$

$$\underline{C}\underline{B}\underline{A} = \begin{pmatrix} 2\sqrt5 & -\frac{1}{\sqrt5} \\ \sqrt5 & \frac{2}{\sqrt5} \end{pmatrix}\begin{pmatrix} \frac{2}{\sqrt5} & \frac{1}{\sqrt5} \\ -\frac{1}{\sqrt5} & \frac{2}{\sqrt5} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{21}{5} & \frac{8}{5} \\ \frac{8}{5} & \frac{9}{5} \end{pmatrix}$$

iii) Stretch $\times \frac{1}{5}$ parallel to $y = \frac{1}{2}x$

First $\underline{A} = \begin{pmatrix} \frac{2}{\sqrt5} & \frac{1}{\sqrt5} \\ -\frac{1}{\sqrt5} & \frac{2}{\sqrt5} \end{pmatrix}$

then $\underline{D}$ stretch $\times \frac{1}{5}$ parallel to x axis

$$\underline{D} = \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & 1 \end{pmatrix}$$

then $\underline{C} = \begin{pmatrix} \frac{2}{\sqrt5} & -\frac{1}{\sqrt5} \\ \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \end{pmatrix}$

Need $\underline{C}\underline{D}\underline{A}$

$$\underline{C}\underline{D} = \begin{pmatrix} \frac{2}{\sqrt5} & -\frac{1}{\sqrt5} \\ \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \end{pmatrix}\begin{pmatrix} \frac{1}{5} & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{5\sqrt5} & -\frac{1}{\sqrt5} \\ \frac{1}{5\sqrt5} & \frac{2}{\sqrt5} \end{pmatrix}$$

$$\underline{C}\underline{D}\underline{A} = \begin{pmatrix} \frac{2}{5\sqrt5} & -\frac{1}{\sqrt5} \\ \frac{1}{5\sqrt5} & \frac{2}{\sqrt5} \end{pmatrix}\begin{pmatrix} \frac{2}{\sqrt5} & \frac{1}{\sqrt5} \\ -\frac{1}{\sqrt5} & \frac{2}{\sqrt5} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{9}{25} & -\frac{8}{25} \\ -\frac{8}{25} & \frac{21}{25} \end{pmatrix}$$