MEI FP1 MATRICES EXERCISE 1H

1 i) $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow$
$$-y = x \quad (1)$$
$$x + 2y = y \quad (2)$$

from (1) $x + y = 0$
from (2) $x + y = 0$

line $x + y = 0$ gives invariant points $(\lambda, -\lambda)$

1 ii)
$$\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow$
$$3x + 4y = x \quad (1)$$
$$x + 2y = y \quad (2)$$

from (1) $2x + 4y = 0$ (3)
from (2) $x + y = 0$ (4)

(3) - 2(4) $2y = 0$
$y = 0$
Sub in (4) $x = 0$

Invariant point is $(0, 0)$

1 iii)
$$\begin{pmatrix} 0{\cdot}6 & 0{\cdot}8 \\ 0{\cdot}8 & -0{\cdot}6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow$
$$0{\cdot}6x + 0{\cdot}8y = x \quad (1)$$
$$0{\cdot}8x - 0{\cdot}6y = y \quad (2)$$

from (1) $-0{\cdot}4x + 0{\cdot}8y = 0$ (3)
from (2) $0{\cdot}8x - 1{\cdot}6y = 0$ (4)
(1)$\times -2$ $0{\cdot}8x - 1{\cdot}6y = 0$

$\Rightarrow$ $x - 2y = 0$
is line of invariant points $(\lambda, \tfrac{1}{2}\lambda)$

1 iv) $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow$
$$\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y = x \quad (1)$$
$$-\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y = y \quad (2)$$

$\Rightarrow$
$$x - y = \sqrt{2}x \quad (3)$$
$$-x + y = \sqrt{2}y \quad (4)$$

from (3) $(\sqrt{2}-1)x + y = 0$ (5)
from (4) $(\sqrt{2}-1)y + x = 0$ (6)
(6)$\times(\sqrt{2}-1)$ $(\sqrt{2}-1)^2y + (\sqrt{2}-1)x = 0$ (7)
(7) - (5) $\left((\sqrt{2}-1)^2 - (\sqrt{2}-1)\right)y = 0$

$\Rightarrow y = 0$
Sub in (4) $\Rightarrow x = 0$

$(0, 0)$ only invariant point

1 v)
$$\begin{pmatrix} 4 & 1 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow$
$$4x + y = x \quad (1)$$
$$6x + 3y = y \quad (2)$$

from (1) $3x + y = 0$
from (2) $6x + 2y = 0$

Line $3x + y = 0$ is line of invariant points $(\lambda, -3\lambda)$

1 vi) $\begin{pmatrix} 7 & -4 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow 7x - 4y = x$ ①

$3x - y = y$ ②

from ① $6x - 4y = 0$

from ② $3x - 2y = 0$

Line $3x - 2y = 0$ is line of invariant points

$(2\lambda, 3\lambda)$

2) $I\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

If $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$ax + cy = x$ ①

$bx + dy = y$ ②

from ① $(a-1)x + cy = 0$ ③

from ② $bx + (d-1)y = 0$ ④

③$\times(d-1)$ $(a-1)(d-1)x + c(d-1)y = 0$

④$\times c$ $bcx + c(d-1)y = 0$

Subtracting

$((a-1)(d-1) - bc)x = 0$

$(ad - d - a + 1 - bc)x = 0$

$(\det T - d - a + 1)x = 0$

If $\det T = a + d - 1$

$(0)x = 0$

True for all x

Similarly

③$\times b$ $(a-1)bx + bcy = 0$

④$\times(a-1)$ $(a-1)bx + (a-1)(d-1)y = 0$

Subtracting $((a-1)(d-1) - bc)y = 0$

$(ad - d - a + 1 - bc)y = 0$

$(\det T - d - a + 1)y = 0$

If $\det T = a + d - 1$

$0y = 0$

True for all y

If $x = \lambda$ say

$(a-1)b\lambda + bcy = 0$

$bcy = (1-a)b\lambda$

$y = \frac{(1-a)\cancel{b}\lambda}{bc}$

$y = \frac{(1-a)\lambda}{c}$

$\therefore$ invariant points other than the origin exist.

3) Let point on $y = mx$ be $(\lambda, \lambda m)$

$$\begin{pmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} \end{pmatrix} \begin{pmatrix} \lambda \\ \lambda m \end{pmatrix}$$

$$= \frac{1}{(1+m^2)} \begin{pmatrix} \lambda(1-m^2) + 2m^2\lambda \\ 2m\lambda + \lambda m(m^2-1) \end{pmatrix}$$

$$= \frac{1}{(1+m^2)} \begin{pmatrix} \lambda - \lambda m^2 + 2\lambda m^2 \\ 2\lambda m + \lambda m^3 - \lambda m \end{pmatrix}$$

$$= \frac{1}{(1+m^2)} \begin{pmatrix} \lambda(1+m^2) \\ \lambda m(1+m) \end{pmatrix}$$

$$= \begin{pmatrix} \lambda \\ \lambda m \end{pmatrix}$$

$\therefore$ all points on $y = mx$ are invariant

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4) i) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

If $\begin{pmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow$ $-0.6x + 0.8y = x$ (1)

$0.8x + 0.6y = y$ (2)

from (1) $-1.6x + 0.8y = 0$ (3)

from (2) $0.8x - 0.4y = 0$ (4)

(3) $\times -\frac{1}{2}$ $0.8x - 0.4y = 0$

Mirror line of invariant points

$$2x - y = 0$$

example point $(1, 2)$

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4 ii)

a) $\begin{pmatrix} X \\ Y \\ 1 \end{pmatrix} = \begin{pmatrix} -0.6 & 0.8 & a \\ 0.8 & 0.6 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$

$$\begin{pmatrix} X \\ Y \\ 1 \end{pmatrix} = \begin{pmatrix} -0.6 & 0.8 & -4 \\ 0.8 & 0.6 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} X \\ Y \\ 1 \end{pmatrix} = \begin{pmatrix} 4 - 4 \\ 0 + 3 + 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix}$$

$(X, Y) = (0, 5)$

so $(0, 5)$ an invariant point.

4 ii) b)

$$\begin{pmatrix} X \\ Y \\ 1 \end{pmatrix} = \begin{pmatrix} -0.6 & 0.8 & 2 \\ 0.8 & 0.6 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} X \\ Y \\ 1 \end{pmatrix} = \begin{pmatrix} -0.6x + 0.8y + 2 \\ 0.8x + 0.6y + 1 \\ 1 \end{pmatrix}$$

For invariant point we need

$$x = -0.6x + 0.8y + 2 \quad (1)$$
$$y = 0.8x + 0.6y + 1 \quad (2)$$

From (1)

$$1.6x - 0.8y = 2 \quad (3)$$

From (2) $\quad 0.4y - 0.8x = 1 \quad (4)$

(4) $\times -2$ $\quad 1.6x - 0.8y = -2 \quad (5)$

But (3) and (5) are now inconsistent

$\therefore$ no solution and no invariant points

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4 ii) c) For an invariant point the following eqns must be consistent

$$1.6x - 0.8y = a$$

$$1.6x - 0.8y = -2b$$

$$\therefore \quad a = -2b$$

or $\quad a + 2b = 0$
