1) i) $\int \frac{1}{4+(x+2)^2}\,dx$

$= \frac{1}{2}\tan^{-1}\frac{(x+2)}{2} + c$

ii) $\int \frac{7}{\sqrt{5+4x-x^2}}\,dx$

$= \int \frac{7}{\sqrt{5-(x^2-4x)}}\,dx$

$= \int \frac{7}{\sqrt{5-((x-2)^2-4)}}\,dx$

$= \int \frac{7}{\sqrt{9-(x-2)^2}}\,dx$

$= 7\sin^{-1}\left(\frac{x-2}{3}\right) + c$

iii) $\int \frac{3}{3+2x^2}\,dx$

$= \int \frac{3}{2\left(\frac{3}{2}+x^2\right)}\,dx$

$= \frac{3}{2}\cdot\frac{1}{\frac{\sqrt{3}}{\sqrt{2}}}\tan^{-1}\left(\frac{x}{\frac{\sqrt{3}}{\sqrt{2}}}\right) + c$

$= \sqrt{\frac{3}{2}}\tan^{-1}\left(\frac{\sqrt{2}\,x}{\sqrt{3}}\right) + c$

iv) $\int \frac{3}{9x^2+6x+5}\,dx$

$= \int \frac{3}{9\left(x^2+\frac{2x}{3}+\frac{5}{9}\right)}\,dx$

$= \frac{1}{3}\int \frac{1}{\left(x+\frac{1}{3}\right)^2+\frac{5}{9}-\frac{1}{9}}\,dx$

$= \frac{1}{3}\int \frac{1}{\left(x+\frac{1}{3}\right)^2+\frac{4}{9}}\,dx$

$= \frac{1}{3}\cdot\frac{1}{\frac{2}{3}}\tan^{-1}\left(\frac{\left(x+\frac{1}{3}\right)}{\frac{2}{3}}\right) + c$

$= \frac{1}{2}\tan^{-1}\left(\frac{3\left(x+\frac{1}{3}\right)}{2}\right) + c$

$= \frac{1}{2}\tan^{-1}\left(\frac{3x+1}{2}\right) + c$

1 vi) $\int \frac{7}{\sqrt{3-4x-4x^2}}\,dx$

$= \int \frac{7}{2\sqrt{3/4-x-x^2}}\,dx$

$= \frac{1}{2}\int \frac{7}{\sqrt{\frac{3}{4}-(x^2+x)}}\,dx$

$= \frac{1}{2}\int \frac{7}{\sqrt{\frac{3}{4}-\left(\left(x+\frac{1}{2}\right)^2-\frac{1}{4}\right)}}$

1 vi) $= \frac{7}{2}\int \frac{1}{\sqrt{1-(x+\frac{1}{2})^2}}\,dx$

$= \frac{7}{2}\sin^{-1}\left(x+\frac{1}{2}\right) + c$

$= \frac{7}{2}\sin^{-1}\left(\frac{2x+1}{2}\right) + c$

---

1 v) $\int \frac{1}{\sqrt{3+2x-x^2}}\,dx$

$= \int \frac{1}{\sqrt{3-(x^2-2x)}}\,dx$

$= \int \frac{1}{\sqrt{3-((x-1)^2-1)}}\,dx$

$= \int \frac{1}{\sqrt{4-(x-1)^2}}\,dx$

$= \sin^{-1}\left(\frac{x-1}{2}\right) + c$

---

2 i) $\int \arcsin x\,dx = \int 1 \arcsin x\,dx$

Let $u = \arcsin x$ Let $\frac{dv}{dx} = 1$

$\Rightarrow \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$ $\Rightarrow v = x$

* $\int \arcsin x\,dx = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,dx$

Now $\int \frac{x}{\sqrt{1-x^2}}\,dx$

Let $u = 1-x^2$

$\Rightarrow \frac{du}{dx} = -2x$

$\Rightarrow -\frac{1}{2}du = x\,dx$

$\therefore \int \frac{x}{\sqrt{1-x^2}}\,dx = \int -\frac{1}{2}\cdot\frac{1}{u^{1/2}}\,du$

$= -\frac{1}{2}\,\frac{u^{1/2}}{\frac{1}{2}} + c$

$= -\sqrt{1-x^2} + c$

From (*)

$\int \arcsin x\,dx = x\arcsin x + \sqrt{1-x^2} + c$

---

2 ii) a) $\int \cos^{-1}x\,dx = \int 1\cos^{-1}x\,dx$

Let $u = \cos^{-1}x$ Let $\frac{dv}{dx} = 1$

$\Rightarrow \frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}}$ $\Rightarrow v = x$

$\int \cos^{-1}x\,dx = x\cos^{-1}x + \int \frac{x}{\sqrt{1-x^2}}\,dx$

$= x\cos^{-1}x - \sqrt{1-x^2} + c$

---

2 ii) b) $\int \arctan x\,dx$

$= \int 1\arctan x\,dx$

2ii b cont) Let $u = \arctan x$ Let $\frac{dv}{dx} = 1$

$\Rightarrow \frac{du}{dx} = \frac{1}{1+x^2}$ $\Rightarrow v = x$

$$\int \arctan x \, dx = x \arctan x - \int \frac{x}{1+x^2} \, dx$$

$$= x \arctan x - \frac{1}{2} \ln(1+x^2) + c$$

2ii c)

$$\int \operatorname{arccot} x \, dx = \int 1 \operatorname{arccot} x \, dx$$

Let $u = \operatorname{arccot} x$ Let $\frac{dv}{dx} = 1$

$\Rightarrow \frac{du}{dx} = -\frac{1}{1+x^2}$ $\Rightarrow v = x$

$$\int \operatorname{arccot} x \, dx = x \operatorname{arccot} x + \int \frac{x}{1+x^2} \, dx$$

$$= x \operatorname{arccot} x + \frac{1}{2} \ln(1+x^2) + c$$

3) i

$$\int_0^b \sqrt{a^2 - x^2} \, dx$$

$a > b > 0$

Let $x = a \sin u$

$\frac{dx}{du} = a \cos u$

$dx = a \cos u \, du$

When $x = b$, $u = \sin^{-1}\left(\frac{b}{a}\right)$

When $x = 0$, $u = \sin^{-1} 0 = 0$

$$\int_0^{\sin^{-1}(b/a)} \left(\sqrt{a^2 - a^2 \sin^2 u}\right) a \cos u \, du$$

$$= \int_0^{\sin^{-1}\left(\frac{b}{a}\right)} a \cos u \times a \cos u \, du$$

$$= \int_0^{\sin^{-1}\left(\frac{b}{a}\right)} a^2 \cos^2 u \, du$$

$$= a^2 \int^{\sin^{-1}\left(\frac{b}{a}\right)} \frac{1}{2}(1 + \cos 2u) \, du$$

$$= a^2 \left[\frac{u}{2} + \frac{1}{4} \sin 2u\right]_0^{\sin^{-1}\frac{b}{a}}$$

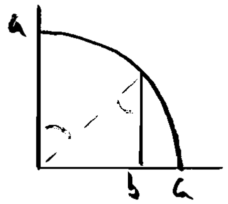
$$= a^2 \left[\frac{u}{2} + \frac{1}{2} \sin u \cos u\right]_0^{\sin^{-1}\left(\frac{b}{a}\right)}$$

$$= a^2 \left[\frac{1}{2} \sin^{-1}\left(\frac{b}{a}\right) + \frac{1}{2} \frac{b}{a} \left(\frac{\sqrt{a^2 - b^2}}{a}\right)\right]$$

$$= \frac{a^2}{2} \sin^{-1}\left(\frac{b}{a}\right) + \frac{a^2 b \sqrt{a^2 - b^2}}{2a^2}$$

$$= \frac{a^2}{2} \sin^{-1}\left(\frac{b}{a}\right) + \frac{b\sqrt{a^2 - b^2}}{2}$$

ii



Area of sector + area of triangle

4 i) $$\int \frac{1}{x^2-6x+13}\,dx$$

$$= \int \frac{1}{(x-3)^2+4}\,dx$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right)+c$$

ii) $$\int \frac{1}{\sqrt{7-12x-4x^2}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{\frac{7}{4}-3x-x^2}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{\frac{7}{4}-(x^2+3x)}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{\frac{7}{4}-\left((x+\frac{3}{2})^2-\frac{9}{4}\right)}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{4-(x+\frac{3}{2})^2}}\,dx$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{x+\frac{3}{2}}{2}\right)+c$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{2x+3}{4}\right)+c$$

iii) $$\int \frac{1}{4x^2+20x+29}\,dx$$

$$= \frac{1}{4}\int \frac{1}{x^2+5x+\frac{29}{4}}\,dx$$

$$= \frac{1}{4}\int \frac{1}{(x+\frac{5}{2})^2+\frac{29}{4}-\frac{25}{4}}\,dx$$

$$= \frac{1}{4}\int \frac{1}{(x+\frac{5}{2})^2+1}\,dx$$

$$= \frac{1}{4}\tan^{-1}(x+\frac{5}{2})+c$$

$$= \frac{1}{4}\tan^{-1}\left(\frac{2x+5}{2}\right)+c$$

iv) $$\int \frac{1}{x^2-6x+9}\,dx$$

$$= \int \frac{1}{(x-3)^2+9-9}\,dx$$

$$= \int \frac{1}{(x-3)^2}\,dx$$

$$= -\frac{1}{x-3}+c$$

v) $$\int \frac{1}{\sqrt{5-12x-9x^2}}\,dx$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{\frac{5}{9}-\frac{12}{9}x-x^2}}\,dx$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{\frac{5}{9}-(x^2+\frac{4}{3}x)}}\,dx$$

1v cont) $= \frac{1}{3}\int \frac{1}{\sqrt{5/9 - (x+\frac{2}{3})^2 - \frac{4}{9}}}\,dx$

$= \frac{1}{3}\int \frac{1}{\sqrt{1 - (x+2/3)^2}}\,dx$

$= \frac{1}{3}\sin^{-1}\left(x+\frac{2}{3}\right) + c$

$= \frac{1}{3}\sin^{-1}\left(\frac{3x+2}{3}\right) + c$

5 i)

$\int \frac{x+1}{x^2+1}\,dx$

$= \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$

$= \frac{1}{2}\ln(x^2+1) + \tan^{-1}x + c$

5ii)

$\int \frac{4}{(x^2+1)(1+x)}\,dx$

Let

$\frac{4}{(x^2+1)(1+x)} \equiv \frac{Ax+B}{x^2+1} + \frac{C}{1+x}$

$\Rightarrow 4 = (Ax+B)(x+1) + C(x^2+1)$

When $x=-1$

$4 = C((-1)^2+1) \quad \Rightarrow C = 2$

When $x=0$

$4 = B(1) + 2(1) \quad \Rightarrow B = 2$

Equating coeffs of x^2

$0 = A + C \quad \Rightarrow A = -2$

$\int \left(\frac{-2x+2}{x^2+1} + \frac{2}{x+1}\right)dx$

$= \int \left(\frac{-2x}{x^2+1} + \frac{2}{x^2+1} + \frac{2}{x+1}\right)dx$

$= -\ln(x^2+1) + 2\tan^{-1}x + 2\ln|x+1| + c$

$= \ln\left|\frac{(x+1)^2}{(x^2+1)}\right| + 2\tan^{-1}x + c$

5iii)

$\int \frac{1-x}{\sqrt{1-x^2}}\,dx$

$= \int \frac{1}{\sqrt{1-x^2}}\,dx - \int \frac{x}{\sqrt{1-x^2}}\,dx$

$= \sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx$

Find $\int \frac{x}{\sqrt{1-x^2}}\,dx$

Let $u = 1-x^2$

$\frac{du}{dx} = -2x$

$-\frac{1}{2}du = x\,dx$

$\int -\frac{1}{2}\frac{1}{u^{1/2}}\,du$

$= -\frac{1}{2}\frac{u^{1/2}}{\frac{1}{2}} + c = -u^{\frac{1}{2}} + c$

$= -\sqrt{1-x^2} + c$

$$\therefore \int \frac{1-x}{\sqrt{1-x^2}}\,dx$$

$$= \sin^{-1}x + \sqrt{1-x^2} + c$$

5iv)

$$\int \frac{x+3}{(x+1)(x^2+1)}\,dx$$

Let $\dfrac{x+3}{(x+1)(x^2+1)} \equiv \dfrac{Ax+B}{(x^2+1)} + \dfrac{C}{(x+1)}$

$$\Rightarrow x+3 \equiv (Ax+B)(x+1) + c(x^2+1)$$

when $x=-1$

$$-1+3 = 2c \quad \Rightarrow c=1$$

when $x=0$

$$3 = B(1) + 1(1) \Rightarrow B=2$$

Equating coeffs of x^2

$$0 = A + c \quad \Rightarrow \quad A=-1$$

$$\int\left(\frac{-x+2}{x^2+1} + \frac{1}{x+1}\right)dx$$

$$= \int\left(-\frac{x}{x^2+1} + \frac{2}{x^2+1} + \frac{1}{x+1}\right)dx$$

$$= -\frac{1}{2}\ln(x^2+1) + 2\tan^{-1}x + \ln|x+1| + c$$

$$= -\frac{1}{2}\ln(x^2+1) + 2\tan^{-1}x + \frac{1}{2}\ln(x+1)^2$$

$$= \frac{1}{2}\ln\left(\frac{(x+1)^2}{(x^2+1)}\right) + 2\tan^{-1}x + c$$

6)i)

$$\int_1^3 \frac{1}{\sqrt{4x-x^2}}\,dx$$

$$= \int_1^3 \frac{1}{\sqrt{0-(x^2-4x)}}\,dx$$

$$= \int_1^3 \frac{1}{\sqrt{0-((x-2)^2-4)}}\,dx$$

$$= \int_1^3 \frac{1}{\sqrt{4-(x-2)^2}}\,dx$$

$$= \left[\sin^{-1}\left(\frac{x-2}{2}\right)\right]_1^3$$

$$= \sin^{-1}\frac{1}{2} - \sin^{-1}\left(-\frac{1}{2}\right)$$

$$= 2\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

6ii)

$$\int_2^5 \frac{2x^2+3}{(x-1)(x^2+4)}\,dx$$

Let $\dfrac{2x^2+3}{(x-1)(x^2+4)} \equiv \dfrac{Ax+B}{x^2+4} + \dfrac{C}{x-1}$

$$\Rightarrow 2x^2+3 \equiv (Ax+B)(x-1) + c(x^2+4)$$

when $x=1$

$$5 = c(5) \quad \Rightarrow c=1$$

when $x=0$

$$3 = B(-1) + 4 \quad \Rightarrow B=1$$

6ii) cont) Equating coeffs of x^2

$$2 = A + C$$
$$2 = A + 1 \quad \Rightarrow A = 1$$

$$\int_2^5 \left(\frac{x+1}{x^2+4} + \frac{1}{x-1}\right) dx$$

$$= \int_2^5 \left(\frac{x}{x^2+4} + \frac{1}{x^2+4} + \frac{1}{x-1}\right) dx$$

$$= \left[\frac{1}{2}\ln(x^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + \ln(x-1)\right]_2^5$$

$$= \left[\frac{1}{2}\ln 29 + \frac{1}{2}\tan^{-1}\left(\frac{5}{2}\right) + \ln 4\right]$$

$$- \left[\frac{1}{2}\ln 8 + \frac{1}{2}\tan^{-1} 1 + \ln 1\right]$$

$$= \frac{1}{2}\ln 29 + \frac{1}{2}\ln 16 + \frac{1}{2}\ln \frac{5}{2}$$

$$- \frac{1}{2}\ln 8 - \frac{1}{2}\cdot\frac{\pi}{4} - 0$$

$$= \frac{1}{2}\ln 58 + \frac{1}{2}\tan^{-1}\frac{5}{2} - \frac{\pi}{8}$$

$$= 2.23 \text{ rad}$$

7) Let $y = \sec^{-1} x$

$$\sec y = x$$

$$\frac{1}{\cos y} = x$$

$$x = (\cos y)^{-1}$$

$$\frac{dx}{dy} = -1(\cos y)^{-2}(-\sin y)$$

$$\frac{dx}{dy} = \frac{\sin y}{\cos^2 y}$$

$$\frac{dy}{dx} = \frac{\cos^2 y}{\sin y}$$

$$\frac{dy}{dx} = \frac{\cos^2 y}{\sqrt{1-\cos^2 y}} = \frac{\frac{1}{x^2}}{\sqrt{1-\frac{1}{x^2}}}$$

$$= \frac{\frac{1}{x^2}}{\sqrt{\frac{x^2-1}{x^2}}} = \frac{\frac{1}{x^2}}{\frac{\sqrt{x^2-1}}{x}}$$

$$= \frac{1}{x\sqrt{x^2-1}}$$

ii) Let $y = \sec^{-1}\left(\frac{x}{a}\right)$

$$\sec y = \frac{x}{a}$$

$$\frac{a}{\cos y} = x$$

$$a(\cos y)^{-1} = \frac{dx}{dy}$$

$$= -a(\cos y)^{-2}(-\sin y) = \frac{dx}{dy}$$

$$\frac{dx}{dy} = \frac{a\sin y}{\cos^2 y}$$

$$\frac{dy}{dx} = \frac{\cos^2 y}{a\sin y} = \frac{\left(\frac{a}{x}\right)^2}{a\sqrt{1-\left(\frac{a}{x}\right)^2}}$$

7 cont

$$\frac{dy}{dx} = \frac{\frac{a^2}{x^2}}{a\sqrt{1-\frac{a^2}{x^2}}}$$

$$= \frac{\frac{a^2}{x^2}}{a\sqrt{\frac{x^2-a^2}{x^2}}}$$

$$= \frac{\frac{a^2}{x^2}}{\frac{a}{x}\sqrt{x^2-a^2}}$$

$$= \frac{a}{x\sqrt{x^2-a^2}}$$

$$\therefore \int \frac{1}{x\sqrt{x^2-a^2}}$$

$$= \frac{1}{a}\operatorname{arc\,sec}\left(\frac{x}{a}\right) + c$$