1) $\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2\,d\theta = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (10\cos\theta)^2\,d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 50\cos^2\theta\,d\theta$

$= 25\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta$

$= 25\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= 25\left[\left(\frac{\pi}{2}+0\right)-\left(-\frac{\pi}{2}+0\right)\right]$

$= 25\pi$

Also $25\int_0^{\pi} (1+\cos 2\theta)\,d\theta$

$= 25\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi}$

$= 25\left[(\pi+0)-(0+0)\right]$

$= 25\pi$

$25\int_0^{2\pi} (1+\cos 2\theta)\,d\theta$

$= 25\left[(2\pi+0)-(0+0)\right]$

$= 50\pi$

If $r = 10\cos\theta$

For cartesian eqn

$x = r\cos\theta = 10\cos^2\theta$

$= 5 + 5\cos 2\theta$

$y = r\sin\theta = 10\cos\theta\sin\theta$

$= 5\sin\theta$

Circle centre $(5,0)$ radius 5

Area $= \pi r^2 = 25\pi$

Circle is traced out for

$0 \le \theta \le \pi$ or $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

for $0 \le \theta \le 2\pi$ circle is traced out twice. Hence integral is doubled

2) $r = \frac{4\theta}{\pi}$

Area $= \frac{1}{2}\int_0^{2\pi} r^2\,d\theta$

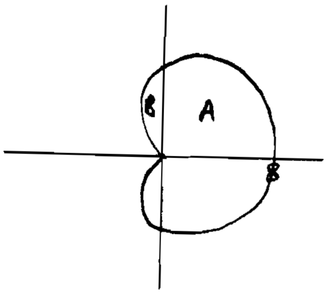
$= \frac{1}{2}\int_0^{2\pi} \frac{16\theta^2}{\pi^2}\,d\theta$

$= \frac{8}{\pi^2}\left[\frac{\theta^3}{3}\right]_0^{2\pi}$

$= \frac{8}{\pi^2}\left[\frac{8\pi^3}{3} - 0\right]$

$= \frac{64\pi}{3}$

3)



$$r = 8(1+\cos\theta)$$

$$\text{Area } A = \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 64(1+\cos\theta)^2\,d\theta$$

$$= 32\int_0^{\frac{\pi}{2}} (1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$= 32\int_0^{\frac{\pi}{2}} \left(1+2\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right)d\theta$$

$$= 32\int_0^{\frac{\pi}{2}} \left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta$$

$$= 32\left[\frac{3\theta}{2}+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= 32\left[\frac{3\pi}{4}+2\right]$$

$$= 24\pi + 64$$

$$\text{Area } B = 32\left[\frac{3\theta}{2}+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_{\frac{\pi}{2}}^{\pi}$$

$$= 32\left[\left(\frac{3\pi}{2}\right)-\left(\frac{3\pi}{4}+2\right)\right]$$

$$= 32\left[\frac{3\pi}{4}-2\right] = 24\pi - 64$$

4) $\text{Area } A = \int_0^{\frac{\pi}{4}} \frac{1}{2} r^2\,d\theta$

$$A = \frac{1}{2}\int_0^{\frac{\pi}{4}} a^2 e^{2k\theta}\,d\theta$$

$$= \frac{a^2}{4k}\left[e^{2k\theta}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{a^2}{4k}\left[e^{\frac{k\pi}{2}}-1\right]$$

$$\text{Area } B = \frac{a^2}{4k}\left[e^{2k\theta}\right]_{2\pi}^{\frac{9\pi}{4}}$$

$$= \frac{a^2}{4k}\left[e^{\frac{9k\pi}{2}}-e^{4k\pi}\right]$$

$$= e^{4k\pi} \times \text{Area } A$$

$$\text{Area } C = \frac{a^2}{4k}\left[e^{2k\theta}\right]_{4\pi}^{\frac{17\pi}{4}}$$

$$= \frac{a^2}{4k}\left[e^{\frac{17k\pi}{2}}-e^{8k\pi}\right]$$

$$= e^{4k\pi} \times \text{Area } B$$

As angles increase by 2π for successive whorls, areas increase by factor of $e^{4k\pi}$

Areas $\therefore$ form a geometric sequence with common ratio $e^{4k\pi}$

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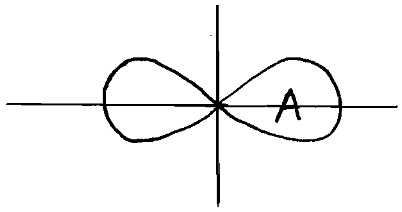
5) $r^2 = a^2\cos 2\theta$ At origin for $\theta = \frac{\pi}{4}$



Area of loop A

$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^2\, d\theta$$

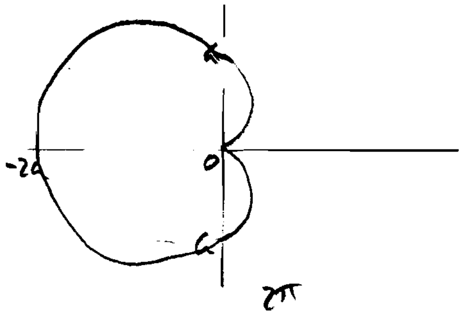
$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} a^2\cos 2\theta$$

$$= \left[\frac{a^2}{4}\sin 2\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{a^2}{4} - -\frac{a^2}{4} = \frac{a^2}{2}$$

6) $r = a(1-\cos\theta)$

i)



$$\text{Area} = \frac{1}{2}\int_0^{2\pi} r^2\, d\theta$$

$$= \frac{1}{2}a^2\int_0^{2\pi}(1-2\cos\theta+\cos^2\theta)\,d\theta$$

$$= \frac{1}{2}a^2\int_0^{2\pi}\left(1-2\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right)d\theta$$

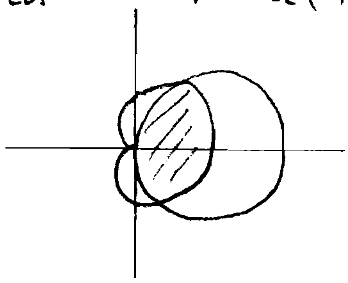
$$= \frac{a^2}{2}\int_0^{2\pi}\left(\frac{3}{2}-2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta$$

$$= \frac{a^2}{2}\left[\frac{3\theta}{2}-2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{2\pi}$$

$$= \frac{a^2}{2}\left[3\pi\right] = \frac{3}{2}a^2\pi$$

7)

$r = 3a\cos\theta$ $\quad r = a(1+\cos\theta)$



Find points of intersection

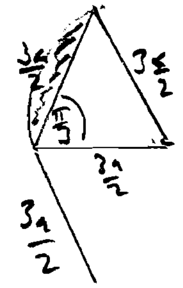
$$3a\cos\theta = a(1+\cos\theta)$$

$$3\cos\theta = 1+\cos\theta$$

$$2\cos\theta = 1$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

$$\theta = -\frac{\pi}{3},\quad r = \frac{3a}{2}$$



Top half of shaded area

$$= \frac{1}{2}\int_0^{\frac{\pi}{3}}\left(a(1+\cos\theta)\right)^2 d\theta \quad + \text{ segment shown}$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{3}}(1+\cos\theta)^2\,d\theta + \frac{1}{2}\left(\frac{3a}{2}\right)^2\left(\frac{\pi}{3}-\sin\frac{\pi}{3}\right)$$

7 cont) $= \frac{a^2}{2}\int_0^{\frac{\pi}{3}} (1+2\cos\theta+\cos^2\theta)\,d\theta$

$+ \frac{1}{2} \times \frac{9a^2}{4}\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right)$

$= \frac{a^2}{2}\int_0^{\frac{\pi}{3}} \left(1+2\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right)d\theta$

$+ \frac{9a^2}{8}\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right)$

$= \frac{a^2}{2}\int_0^{\frac{\pi}{3}} \left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta$

$+ \frac{9a^2}{8}\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right)$

$= \frac{a^2}{2}\left[\frac{3\theta}{2}+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{3}}$

$+ \frac{9a^2}{8}\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right)$

$= \frac{a^2}{2}\left[\left(\frac{\pi}{2}+\sqrt{3}+\frac{\sqrt{3}}{8}\right)\right]$

$+ \frac{9a^2}{8}\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right)$

$= \frac{a^2}{2}\left[\frac{\pi}{2}+\frac{9\sqrt{3}}{8}\right]$

$+ \frac{9a^2}{8}\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right)$

$= a^2\left[\frac{\pi}{4}+\cancel{\frac{9\sqrt{3}}{16}}\right]$

$+ a^2\left(\frac{9\pi}{24}-\cancel{\frac{9\sqrt{3}}{16}}\right)$

$= a^2\left(\frac{\pi}{4}+\frac{9\pi}{24}\right)$

$= a^2\left(\frac{6\pi}{24}+\frac{9\pi}{24}\right)$

$= a^2 \times \frac{15\pi}{24}$

$= \frac{5a^2\pi}{8}$

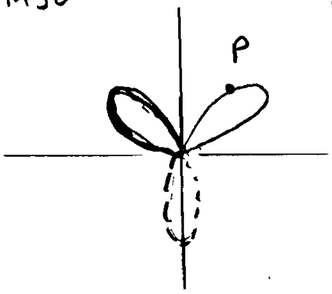
$\therefore$ Top and Bottom halves of shaded area

$= 2 \times \frac{5a^2\pi}{8}$

$= \frac{5a^2\pi}{4}$

12) $r = a\sin 3\theta$ $\quad 0 \le \theta \le \pi$

i)



ii) Area for 1 loop

$$= \frac{1}{2}\int_0^{\frac{\pi}{3}} r^2\, d\theta$$

$$= \frac{1}{2}a^2 \int_0^{\frac{\pi}{3}} \sin^2 3\theta\, d\theta$$

$$= \frac{1}{4}a^2 \int_0^{\frac{\pi}{3}} (1-\cos 6\theta)\, d\theta$$

$$= \frac{a^2}{4}\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\frac{\pi}{3}}$$

$$= \frac{a^2}{4}\left[\frac{\pi}{3}\right] = \frac{\pi a^2}{12}$$

iii)

Polar coords of P are $\left(a\sin\frac{3\pi}{4}, \frac{\pi}{4}\right)$

$$= \left(\frac{a}{\sqrt{2}}, \frac{\pi}{4}\right)$$

$$= \left(\frac{\sqrt{2}a}{2}, \frac{\pi}{4}\right)$$

Cartesian coords

$$x = r\cos\theta = \frac{a}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{a}{2}$$

$$y = r\sin\theta = \frac{a}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{a}{2}$$

P is $\left(\frac{a}{2}, \frac{a}{2}\right)$

iv)

$$x^4 + 2x^2y^2 + y^4 = 3ax^2y - ay^3$$

$$4x^3 + 2x^2 2y\frac{dy}{dx} + 4xy^2 + 4y^3\frac{dy}{dx}$$

$$= 3ax^2\frac{dy}{dx} + 6axy - 3ay^2\frac{dy}{dx}$$

$$\left(4x^2y + 4y^3 - 3ax^2 + 3ay^2\right)\frac{dy}{dx}$$

$$= 6axy - 4x^3 - 4xy^2$$

$$\frac{dy}{dx} = \frac{6axy - 4x^3 - 4xy^2}{4x^2y + 4y^3 - 3ax^2 + 3ay^2}$$

At $\left(\frac{a}{2}, \frac{a}{2}\right)$

$$\frac{dy}{dx} = \frac{\frac{6a^3}{4} - \frac{4a^3}{8} - \frac{4a^3}{8}}{\frac{4a^3}{8} + \frac{4a^3}{8} - \cancel{\frac{3a^3}{4}} + \cancel{\frac{3a^3}{4}}}$$

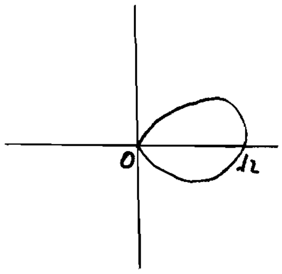
$$\frac{dy}{dx} = \frac{\frac{4a^3}{8}}{\frac{8a^3}{8}} = \frac{1}{2}$$

Gradient at P $= \frac{1}{2}$

13) $r = 2\sqrt{\cos 2\theta}$

$$-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$$

i)



ii) Area $= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^2 \, d\theta$

$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 4\cos 2\theta \, d\theta$$

$$= \left[\sin 2\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= 1 - -1 = 2$$

iii)

$$r^2 = 4(\cos^2\theta - \sin^2\theta)$$

Now $r^2 = x^2 + y^2$

and $x = r\cos\theta$, $y = r\sin\theta$

$\therefore \cos\theta = \frac{x}{r}$ $\quad \sin\theta = \frac{y}{r}$

$$\Rightarrow r^2 = 4\left(\frac{x^2}{r^2} - \frac{y^2}{r^2}\right)$$

$$r^2 \times r^2 = 4(x^2 - y^2)$$

$$(x^2+y^2)^2 = 4(x^2-y^2)$$

$$x^4 + 2x^2y^2 + y^4 = 4(x^2 - y^2)$$

$$x^4 + 2x^2y^2 + y^4 - 4x^2 + 4y^2 = 0$$

iv)

$$4x^3 + 4x^2y\frac{dy}{dx} + 4xy^2 + 4y^3\frac{dy}{dx}$$

$$-8x + 8y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(4x^2y + 4y^3 + 8y\right) = 8x - 4x^3 - 4xy^2$$

$$\frac{dy}{dx} = \frac{8x - 4x^3 - 4xy^2}{4x^2y + 4y^3 + 8y}$$

$$\frac{dy}{dx} = \frac{2x - x^3 - xy^2}{x^2y + y^3 + 2y}$$

v) If $\frac{dy}{dx} = 0$ then $2x - x^3 - xy^2 = 0$

$$xy^2 = 2x - x^3$$

$$y^2 = 2 - x^2$$

$$x^2 + y^2 = 2$$

$\therefore P(x,y)$ on circle centre $(0,0)$

radius $\sqrt{2}$ $\quad \therefore OP = \sqrt{2}$

vi)

$\frac{dy}{dx} \Rightarrow x^2 + y^2 = 2$

$\Rightarrow r^2 = 2$

$\Rightarrow r = \sqrt{2}$

Since $r = 2\sqrt{\cos 2\theta}$

$$\sqrt{2} = 2\sqrt{\cos 2\theta}$$

$$\frac{\sqrt{2}}{2} = \sqrt{\cos 2\theta}$$

$$\frac{1}{\sqrt{2}} = \sqrt{\cos 2\theta}$$

$$\frac{1}{2} = \cos 2\theta$$

$$\Rightarrow 2\theta = \frac{\pi}{3} \text{ or } -\frac{\pi}{3}$$

$$\theta = \frac{\pi}{6} \text{ or } -\frac{\pi}{6}$$

Polar coords of points where $\frac{dy}{dx} = 0$

$$\left(\sqrt{2}, \frac{\pi}{6}\right) \quad \left(\sqrt{2}, -\frac{\pi}{6}\right)$$