1) $\cos 4\theta + j\sin 4\theta = (\cos\theta + j\sin\theta)^4$

Let $c = \cos\theta$, $s = \sin\theta$

$= c^4 + 4c^3js + 6c^2j^2s^2 + 4cj^3s^3 + j^4s^4$

$= c^4 + 4c^3js - 6c^2s^2 - 4cjs^3 + s^4$

Equating real and imaginary parts

$\cos 4\theta = c^4 - 6c^2s^2 + s^4$

$\sin 4\theta = 4c^3s - 4cs^3$

$$\tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta}$$

$$= \frac{4c^3s - 4cs^3}{c^4 - 6c^2s^2 + s^4}$$

Divide throughout by c^4

$$= \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

2) $\cos 3\theta + j\sin 3\theta = (\cos\theta + j\sin\theta)^3$

$= c^3 + 3c^2js + 3cj^2s^2 + j^3s^3$

$= c^3 + 3c^2sj - 3cs^2 - s^3j$

Equating real and imaginary parts

$\cos 3\theta = c^3 - 3cs^2$

$\sin 3\theta = 3c^2s - s^3$

i) $\cos 3\theta = c^3 - 3c(1 - c^2)$

$= c^3 - 3c + 3c^3$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

ii)

$\sin 3\theta = 3(1 - s^2)s - s^3$

$= 3s - 3s^3 - s^3$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

iii)

$$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta}$$

$$\tan 3\theta = \frac{3c^2s - s^3}{c^3 - 3cs^2}$$

Dividing throughout by c^3

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

3)

$\cos 6\theta + j\sin 6\theta = (\cos\theta + j\sin\theta)^6$

$= c^6 + 6c^5js + 15c^4j^2s^2 + 20c^3j^3s^3$

$+ 15c^2j^4s^4 + 6cj^5s^5 + j^6s^6$

$= c^6 + 6c^5sj - 15c^4s^2 - 20c^3s^3j$

$+ 15c^2s^4 + 6cs^5j - s^6$

Equating real and imaginary parts

$\cos 6\theta = c^6 - 15c^4s^2 + 15c^2s^4 - s^6$

$= c^6 - 15c^4(1 - c^2) + 15c^2(1 - c^2)^2 - (1 - c^2)^3$

$= c^6 - 15c^4 + 15c^6 + 15c^2(1 - 2c^2 + c^4)$

$- (1 - 3c^2 + 3c^4 - c^6)$

$= c^6 - 15c^4 + 15c^6 + 15c^2 - 30c^4 + 15c^6$

$- 1 + 3c^2 - 3c^4 + c^6$

3 cont) $= 32c^6 - 48c^4 + 18c^2 - 1$

$= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$

$\sin 6\theta = 6c^5s - 20c^3s^3 + 6cs^5$

$\frac{\sin 6\theta}{\sin\theta} = 6c^5 - 20c^3(1-c^2) + 6c(1-c^2)^2$

$= 6c^5 - 20c^3 + 20c^5 + 6c(1 - 2c^2 + c^4)$

$= 6c^5 - 20c^3 + 20c^5 + 6c - 12c^3 + 6c^5$

$= 32c^5 - 32c^3 + 6c$

$= 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta$

---

4) $\cos n\theta + j\sin n\theta = (\cos\theta + j\sin\theta)^n$

$= c^n + {}^nC_1 c^{n-1} js + {}^nC_2 c^{n-2} j^2 s^2$

$+ {}^nC_3 c^{n-3} j^3 s^3 + {}^nC_4 c^{n-4} j^4 s^4 + \dots$

$= c^n + {}^nC_1 c^{n-1} sj - {}^nC_2 c^{n-2} s^2$

$- {}^nC_3 c^{n-3} s^3 j + {}^nC_4 c^{n-4} s^4 + \dots$

Equating real and imaginary parts

$\cos(n\theta) = c^n - {}^nC_2 c^{n-2} s^2 + {}^nC_4 c^{n-4} s^4 - \dots$

$= c^n\left(1 - {}^nC_2 t^2 + {}^nC_4 t^4 - + \dots\right)$

$\sin(n\theta) = {}^nC_1 c^{n-1} s - {}^nC_3 c^{n-3} s^3 + \dots$

$= c^n\left({}^nC_1 t - {}^nC_3 t^3 + - \dots\right)$

$\therefore \tan(n\theta) = \frac{c^n({}^nC_1 t - {}^nC_3 t^3 + - \dots)}{c^n(1 - {}^nC_2 t^2 + {}^nC_4 t^4 - + \dots)}$

$\tan(n\theta) = \frac{({}^nC_1 t - {}^nC_3 t^3 + - \dots)}{(1 - {}^nC_2 t^2 + {}^nC_4 t^4 - + \dots)}$

---

5) Using $\cos n\theta = \frac{z^n + z^{-n}}{2}$

$\sin n\theta = \frac{z^n - z^{-n}}{2j}$

i) Find $\cos^4\theta$

$\cos^4\theta = \left(\frac{z^1 + z^{-1}}{2}\right)^4$

$= \frac{1}{16}\left[z^4 + 4z^3z^{-1} + 6z^2z^{-2} + 4zz^{-3} + z^{-4}\right]$

$= \frac{1}{16}\left[(z^4 + z^{-4}) + 4(z^2 + z^{-2}) + 6\right]$

$= \frac{1}{16}\left[2\cos 4\theta + 8\cos 2\theta + 6\right]$

$= \frac{\cos 4\theta + 4\cos 2\theta + 3}{8}$

---

ii) Find $\sin^5\theta$

$\sin^5\theta = \left(\frac{z^1 - z^{-1}}{2j}\right)^5$

$= \frac{1}{32j}\left[z^5 - 5z^4z^{-1} + 10z^3z^{-2} - 10z^2z^{-3} + 5zz^{-4} - z^{-5}\right]$

$= \frac{1}{32j}\left[(z^5 - z^{-5}) - 5(z^3 - z^{-3}) + 10(z^1 - z^{-1})\right]$

$= \frac{1}{32j}\left[2j\sin 5\theta - 10j\sin 3\theta + 20j\sin\theta\right]$

$= \frac{\sin 5\theta - 5\sin 3\theta + 10\sin\theta}{16}$

5iii) Find $\sin^6\theta$

$\sin^6\theta = \left(\frac{z^1-z^{-1}}{2j}\right)^6$

$= -\frac{1}{64}\left[z^6 - 6z^5z^{-1} + 15z^4z^{-2} - 20z^3z^{-3} + 15z^2z^{-4} - 6zz^{-5} + z^{-6}\right]$

$= -\frac{1}{64}\left[(z^6+z^{-6}) - 6(z^4+z^{-4}) + 15(z^2+z^{-2}) - 20\right]$

$= -\frac{1}{64}\left[2\cos6\theta - 12\cos4\theta + 30\cos2\theta - 20\right]$

$= -\frac{\cos6\theta - 6\cos4\theta + 15\cos2\theta - 10}{32}$

$= \frac{-\cos6\theta + 6\cos4\theta - 15\cos2\theta + 10}{32}$

5iv) $\cos^3\theta\sin^4\theta$

$= \cos^3\theta\left(1-\cos^2\theta\right)^2$

$= \cos^3\theta\left(1 - 2\cos^2\theta + \cos^4\theta\right)$

$= \cos^3\theta - 2\cos^5\theta + \cos^7\theta$

Find $\cos^3\theta$

$\cos^3\theta = \left(\frac{z^1+z^{-1}}{2}\right)^3$

$= \frac{1}{8}\left[z^3 + 3z^2z^{-1} + 3z^1z^{-2} + z^{-3}\right]$

$= \frac{1}{8}\left[(z^3+z^{-3}) + 3(z^1+z^{-1})\right]$

$= \frac{1}{8}\left[2\cos3\theta + 6\cos\theta\right]$

Find $\cos^5\theta$

$\cos^5\theta = \left(\frac{z^1+z^{-1}}{2}\right)^5$

$= \frac{1}{32}\left[z^5 + 5z^4z^{-1} + 10z^3z^{-2} + 10z^2z^{-3} + 5z^1z^{-4} + z^{-5}\right]$

$= \frac{1}{32}\left[(z^5+z^{-5}) + 5(z^3+z^{-3}) + 10(z^1+z^{-1})\right]$

$= \frac{1}{32}\left[2\cos5\theta + 10\cos3\theta + 20\cos\theta\right]$

Find $\cos^7\theta$

$\cos^7\theta = \left(\frac{z^1+z^{-1}}{2}\right)^7$

$= \frac{1}{128}\left[z^7 + 7z^6z^{-1} + 21z^5z^{-2} + 35z^4z^{-3} + 35z^3z^{-4} + 21z^2z^{-5} + 7z^1z^{-6} + z^{-7}\right]$

$= \frac{1}{128}\left[(z^7+z^{-7}) + 7(z^5+z^{-5}) + 21(z^3+z^{-3}) + 35(z^1+z^{-1})\right]$

$= \frac{1}{128}\left[2\cos7\theta + 14\cos5\theta + 42\cos3\theta + 70\cos\theta\right]$

5 iv) cont

$$\therefore \cos^3\theta - 2\cos^5\theta + \cos^7\theta =$$

$$\frac{1}{8}\left[2\cos 3\theta + 6\cos\theta\right]$$

$$-\frac{2}{32}\left[2\cos 5\theta + 10\cos 3\theta + 20\cos\theta\right]$$

$$+\frac{1}{128}\left[2\cos 7\theta + 14\cos 5\theta + 42\cos 3\theta + 70\cos\theta\right]$$

$$= \frac{1}{64}\left[16\cos 3\theta + 48\cos\theta\right]$$

$$-\frac{1}{64}\left[8\cos 5\theta + 40\cos 3\theta + 80\cos\theta\right]$$

$$+\frac{1}{64}\left[\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta\right]$$

$$= \frac{1}{64}\left[\cos 7\theta - \cos 5\theta - 3\cos 3\theta + 3\cos\theta\right]$$

$$= \frac{\cos 7\theta - \cos 5\theta - 3\cos 3\theta + 3\cos\theta}{64}$$

7) i)

$$\int \sin^6\theta \, d\theta$$

$$= \int \left(\frac{-\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10}{32}\right) d\theta$$

$$= \frac{1}{32}\left(-\frac{1}{6}\sin 6\theta + \frac{6}{4}\sin 4\theta - \frac{15}{2}\sin 2\theta + 10\theta\right) + c$$

$$= \frac{-\sin 6\theta + 9\cos 4\theta - 45\sin 2\theta + 60\theta}{192} + c$$

7 ii)

$$\int_0^{\frac{\pi}{2}} \cos^3\theta \sin^4\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{\cos 7\theta - \cos 5\theta - 3\cos 3\theta + 3\cos\theta}{64}\right) d\theta$$

$$= \frac{1}{64}\left[\frac{\sin 7\theta}{7} - \frac{\sin 5\theta}{5} - \frac{3\sin 3\theta}{3} + 3\sin\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{64}\left[-\frac{1}{7} - \frac{1}{5} + 1 + 3\right]$$

$$= \frac{2}{35}$$

5 v)

$$\cos^4\theta \sin^3\theta$$

$$= (1 - \sin^2\theta)^2 \sin^3\theta$$

$$= (1 - 2\sin^2\theta + \sin^4\theta)\sin^3\theta$$

$$= \sin^3\theta - 2\sin^5\theta + \sin^7\theta$$

Find $\sin^3\theta$

$$\sin^3\theta = \left(\frac{z^1 - z^{-1}}{2j}\right)^3$$

$$= -\frac{1}{8j}\left(z^3 - 3z^2z^{-1} + 3zz^{-2} - z^{-3}\right)$$

$$= -\frac{1}{8j}\left((z^3 - z^{-3}) - 3(z^1 - z^{-1})\right)$$

$$= -\frac{1}{8j}\left(2j\sin 3\theta - 6j\sin\theta\right)$$

$$= -\frac{1}{4}\sin 3\theta + \frac{3}{4}\sin\theta$$

5v cont) Find $\sin^5\theta$

$$\sin^5\theta = \left(\frac{z^1 - z^{-1}}{2j}\right)^5$$

$$= \frac{1}{32j}\left(z^5 - 5z^4z^{-1} + 10z^3z^{-2} - 10z^2z^{-3} + 5zz^{-4} - z^{-5}\right)$$

$$= \frac{1}{32j}\left((z^5 - z^{-5}) - 5(z^3 - z^{-3}) + 10(z^1 - z^{-1})\right)$$

$$= \frac{1}{32j}\left(2j\sin 5\theta - 10j\sin 3\theta + 20j\sin\theta\right)$$

$$= \frac{\sin 5\theta - 5\sin 3\theta + 10\sin\theta}{16}$$

Find $\sin^7\theta$

$$\sin^7\theta = \left(\frac{z^1 - z^{-1}}{2j}\right)^7$$

$$= -\frac{1}{128j}\left(z^7 - 7z^6z^{-1} + 21z^5z^{-2} - 35z^4z^{-3} + 35z^3z^{-4} - 21z^2z^{-5} + 7zz^{-6} - z^{-7}\right)$$

$$= \frac{-1}{128j}\left((z^7 - z^{-7}) - 7(z^5 - z^{-5}) + 21(z^3 - z^{-3}) - 35(z^1 - z^{-1})\right)$$

$$= \frac{-1}{128j}\left(2j\sin 7\theta - 14j\sin 5\theta + 42j\sin 3\theta - 70j\sin\theta\right)$$

$$= \frac{-\sin 7\theta + 7\sin 5\theta - 21\sin 3\theta + 35\sin\theta}{64}$$

$\therefore \sin^3\theta - 2\sin^5\theta + \sin^7\theta =$

$$\frac{-\sin 3\theta + 3\sin\theta}{4} + \frac{-\sin 5\theta + 5\sin 3\theta - 10\sin\theta}{8} + \frac{-\sin 7\theta + 7\sin 5\theta - 21\sin 3\theta + 35\sin\theta}{64}$$

$$= \frac{1}{64}\left(-\sin 7\theta - \sin 5\theta + 3\sin 3\theta + 3\sin\theta\right)$$

7iii)
$$\int_0^\pi \cos^4\theta \sin^3\theta \, d\theta$$

$$= \frac{1}{64}\int_0^\pi \left(-\sin 7\theta - \sin 5\theta + 3\sin 3\theta + 3\sin\theta\right) d\theta$$

$$= \frac{1}{64}\left[\frac{\cos 7\theta}{7} + \frac{\cos 5\theta}{5} - \cos 3\theta - 3\cos\theta\right]_0^\pi$$

$$= \frac{1}{64}\left[\left(-\frac{1}{7} - \frac{1}{5} + 1 + 3\right) - \left(\frac{1}{7} + \frac{1}{5} - 1 - 3\right)\right]$$

$$= \frac{1}{64}\left[-\frac{2}{7} - \frac{2}{5} + 2 + 6\right]$$

$$= \frac{4}{35}$$

8)
$$\cos n\theta = \frac{z^n + z^{-n}}{2}$$

$\therefore \cos\theta + \cos 3\theta + \dots + \cos(2n-1)\theta$

$$= \frac{z^1 + z^{-1}}{2} + \frac{z^3 + z^{-3}}{2} + \dots + \frac{z^{2n-1} + z^{-(2n-1)}}{2}$$

8 cont) $= \frac{1}{2}\left[z^{-(2n-1)} + z^{-(2n-1)+2} + \dots + z^{-1} + z^{1} + z^{3} + \dots + z^{2n-1} \right]$

Within brackets is a GP

$a = z^{-(2n-1)}$, $r = z^2$, $n' = 2n$

Sum

$$\frac{1}{2}\left[\frac{z^{-(2n-1)}\left(z^{4n}-1\right)}{z^2-1} \right]$$

$$= \frac{1}{2}\left[\frac{z^{2n+1} - z^{-(2n-1)}}{z^2-1} \right]$$

Divide top and bottom by z

$$= \frac{1}{2}\left[\frac{z^{2n} - z^{-2n}}{z - z^{-1}} \right]$$

$$= \frac{1}{2}\left[\frac{\frac{z^{2n} - z^{-2n}}{2j}}{\frac{z^{1} - z^{-1}}{2j}} \right]$$

$$= \frac{1}{2}\left[\frac{\sin 2n\theta}{\sin\theta} \right]$$

$$= \frac{\sin 2n\theta}{2\sin\theta}$$

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9)

i) $z = \cos\theta + j\sin\theta$

$\Rightarrow z^n = \cos n\theta + j\sin n\theta$

$z^{-n} = \cos(-n\theta) + j\sin(-n\theta)$

$= \cos n\theta - j\sin n\theta$

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

$$z^n - \frac{1}{z^n} = 2j\sin n\theta$$

---

9 ii)

$z^5 = (c + js)^5$

where $c = \cos\theta$, $s = \sin\theta$

$= c^5 + 5c^4js - 10c^3s^2 - 10c^2js^3 + 5cs^4 + js^5$

But $z^5 = \cos 5\theta + j\sin 5\theta$

Equating real and imaginary parts

$\cos 5\theta = c^5 - 10c^3s^2 + 5cs^4$

$\sin 5\theta = 5c^4s - 10c^2s^3 + s^5$

$\therefore \tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta}$

$$= \frac{5c^4s - 10c^2s^3 + s^5}{c^5 - 10c^3s^2 + 5cs^4}$$

Dividing throughout by c^5

$$= \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$

---

9 iii)

$$\left(z - \frac{1}{z}\right)^2\left(z + \frac{1}{z}\right)^4$$

$$= (2j\sin\theta)^2(2\cos\theta)^4$$

$$= -64\sin^2\theta\cos^4\theta$$

9iii) cont)

$$\left(z-\frac{1}{z}\right)^2\left(z+\frac{1}{z}\right)^4$$

$$=\left(z^2-2+\frac{1}{z^2}\right)\left(z^4+4z^2+6+4z^{-2}+z^{-4}\right)$$

$$\begin{aligned}= z^6 &- 2z^4 + z^2\\ &+4z^4-8z^2+4\\ &+6z^2-12+6z^{-2}\\ &+4-8z^{-2}+4z^{-4}\\ &+z^{-2}-2z^{-4}+z^{-6}\end{aligned}$$

$$= z^6+2z^4-z^2-4-z^{-2}+2z^{-4}+z^{-6}$$

$$=\left(z^6+z^{-6}\right)+2\left(z^4+z^{-4}\right)-\left(z^2+z^{-2}\right)-4$$

$$= 2\cos 6\theta+4\cos 4\theta-2\cos 2\theta-4$$

$$\therefore \quad \sin^2\theta\cos^4\theta =$$

$$\frac{-1}{64}\left(-4-2\cos 2\theta+4\cos 4\theta+2\cos 6\theta\right)$$

$$=\frac{1}{16}+\frac{1}{32}\cos 2\theta-\frac{1}{16}\cos 4\theta-\frac{1}{32}\cos 6\theta$$

10)

i)

$$z=\cos\theta+j\sin\theta$$

$$z^n=\cos n\theta+j\sin n\theta$$

$$z^{-n}=\cos n\theta-j\sin n\theta$$

$$z^n+\frac{1}{z^n}=2\cos n\theta$$

$$z^n-\frac{1}{z^n}=2j\sin n\theta$$

10ii)

$$\left(z-\frac{1}{z}\right)^4\left(z+\frac{1}{z}\right)^2$$

$$=\left(z^4-4z^3z^{-1}+6z^2z^{-2}-4zz^{-3}+z^{-4}\right)\times\left(z^2+2z^1z^{-1}+z^{-2}\right)$$

$$=\left(z^4-4z^2+6-4z^{-2}+z^{-4}\right)\times\left(z^2+2+z^{-2}\right)$$

$$\begin{aligned}= z^6 &-4z^4+6z^2-4+z^{-2}\\ &+2z^4-8z^2+12-8z^{-2}+2z^{-4}\\ &+z^2-4+6z^{-2}-4z^{-4}+z^{-6}\end{aligned}$$

$$=\left(z^6+z^{-6}\right)-2\left(z^4+z^{-4}\right)-\left(z^2+z^{-2}\right)+4$$

$$=2\cos 6\theta-4\cos 4\theta-2\cos 2\theta+4$$

$$\therefore\ (2j\sin\theta)^4(2\cos\theta)^2=64\sin^4\theta\cos^2\theta$$

$$=2\cos 6\theta-4\cos 4\theta-2\cos 2\theta+4$$

$$\therefore\ \sin^4\theta\cos^2\theta=\frac{\cos 6\theta-2\cos 4\theta-\cos 2\theta+2}{32}$$

10iii)

$$\int_1^2 x^4\sqrt{4-x^2}\,dx$$

Let $x=2\sin\theta$

$$\frac{dx}{d\theta}=2\cos\theta$$

$$dx=2\cos\theta\,d\theta$$

when $x=2$, $\theta=\frac{\pi}{2}$

when $x=1$, $\theta=\frac{\pi}{6}$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}16\sin^4\theta\times 2\cos\theta\times 2\cos\theta\,d\theta$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}64\sin^4\theta\cos^2\theta\,d\theta$$

10iii) cont)

$$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left(\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2\right)$$

$$= 2\left[\frac{1}{6}\sin 6\theta - \frac{1}{2}\sin 4\theta - \frac{1}{2}\sin 2\theta + 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= 2\left[\left(0 - 0 - 0 + \pi\right) - \left(0 - \frac{1}{2}\sin\frac{2\pi}{3} - \frac{1}{2}\sin\frac{\pi}{3} + \frac{\pi}{3}\right)\right]$$

$$= 2\left[\pi + \frac{1}{2}\times\frac{\sqrt{3}}{2} + \frac{1}{2}\times\frac{\sqrt{3}}{2} - \frac{\pi}{3}\right]$$

$$= 2\left[\frac{2\pi}{3} + \frac{\sqrt{3}}{2}\right]$$

$$= \frac{4\pi}{3} + \sqrt{3}$$