1) The fifth roots of unity give alternate tenth roots, and their negatives (given by half turns about 0) fill the gaps

2)

i) $(w+1)(1+w^2)$

$= w + 1 + w^3 + w^2$

$= 1 + w + w^2 + 1$

But $1 + w + w^2$

$= \frac{a(1-r^n)}{1-r} = \frac{1(1-w^3)}{1-w} = 0$

$\therefore (w+1)(1+w^2) = 1$

ii) $(1+w)^3 = 1 + 3w + 3w^2 + w^3$

$= 3 + 3w + 3w^2 + w^3 - 2$

$= 3(1+w+w^2) + 1 - 2$

$= 0 + 1 - 2 = -1$

$\therefore$ $1+w$ is cubic root of -1

$(1+w^2)^3 = 1 + 3w^2 + 3w^4 + w^6$

$= 3(1+w^2+w^4) + 1 - 2$

$= 3\left(\frac{1-(w^2)^3}{1-w^2}\right) - 1$

$= 3\left(\frac{1-w^6}{1-w^2}\right) - 1$

$= 0 - 1 = -1$

$\therefore$ $1+w^2$ is cubic root of -1

iii) $(a+b)(a+wb)(a+w^2b)$

$= (a^2 + ab + wab + wb^2)(a+w^2b)$

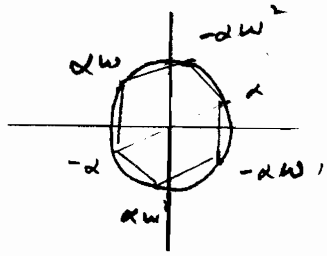
$= a^3 + a^2b + wa^2b + wab^2 + w^2a^2b + w^2ab^2 + w^3ab^2 + w^3b^3$

$= a^3 + a^2b(1+w+w^2) + ab^2(1+w+w^2) + b^3$

$= a^3 + 0 + 0 + b^3 = a^3 + b^3$

iv) Too tedious!

3)



$-\alpha, \pm\alpha w, \pm\alpha w^2$

6) $z^3 = (j-z)^3$

$$\left(\frac{z}{j-z}\right)^3 = 1$$

$$\Rightarrow \frac{z}{j-z} = \alpha \quad \text{where } \alpha = e^{j\theta}$$

$$z = \alpha(j-z)$$

$$z = \alpha j - \alpha z$$

$$z + \alpha z = \alpha j$$

$$z(1+\alpha) = \alpha j$$

$$z = \frac{\alpha j}{1+\alpha}$$

$$z = \frac{e^{j\theta} j}{2\cos\frac{\theta}{2} e^{j\frac{\theta}{2}}}$$

$$z = \frac{e^{j\frac{\theta}{2}} j}{2\cos\frac{\theta}{2}}$$

$$z = \frac{\left(\cos\frac{\theta}{2} + j\sin\frac{\theta}{2}\right)j}{2\cos\frac{\theta}{2}}$$

$$z = \frac{j}{2}\left(1 + j\tan\theta\right)$$

$$z = \frac{j}{2} - \frac{1}{2}\tan\theta$$

where $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$

$$z = \frac{j}{2}, \ \frac{j+\sqrt{3}}{2}, \ \frac{j-\sqrt{3}}{2}$$

7) $z^5 + z^4 + z^3 + z^2 + z + 1 = 0$

$$\frac{z^6 - 1}{z - 1} = 0$$

$$\Rightarrow z^6 = 1$$

$$z = \cos\theta + j\sin\theta$$

for $\theta = \frac{2k\pi}{6}$ $\quad k = \cancel{0}, 1, 2, 3, 4, 5$

However, $k = 0$ does not give a root $\frac{0}{0}$

$$\therefore z = \cos\frac{k\pi}{3} + j\sin\frac{k\pi}{3}$$

for $k = 1, 2, 3, 4, 5$

8) $(z-1)^n = z^n$

$$\left(\frac{z-1}{z}\right)^n = 1$$

$$\frac{z-1}{z} = \alpha \quad \text{where } \alpha = e^{j\theta}$$

$$z - 1 = \alpha z$$

$$z - \alpha z = 1$$

$$z(1-\alpha) = 1$$

$$z = \frac{1}{1-\alpha}$$

$$z = \frac{1}{-2j\sin\frac{\theta}{2} e^{j\frac{\theta}{2}}}$$

$$z = \frac{e^{-j\frac{\theta}{2}}}{-2j\sin\frac{\theta}{2}}$$

$$z = \frac{\cos\frac{\theta}{2} - j\sin\frac{\theta}{2}}{-2j\sin\frac{\theta}{2}}$$

8 cont

$$z = \frac{1}{2} + \frac{1}{2} j \cot\frac{\theta}{2}$$

for $\theta = \frac{2\pi k}{n}$ $(k = 0, 1, 2, \ldots n-1)$

$\therefore$ all roots have real part $= \frac{1}{2}$

9)

$$(j - z)^n = (jz - 1)^n$$

$$\left(\frac{j - z}{jz - 1}\right)^n = 1$$

$$\frac{j - z}{jz - 1} = \alpha \quad \text{where } \alpha = e^{j\theta}$$

$$j - z = \alpha j z - \alpha$$

$$j + \alpha = \alpha j z + z$$

$$j + \alpha = z(\alpha j + 1)$$

$$z = \frac{j + \alpha}{1 + \alpha j}$$

$$z = \frac{j + \alpha}{1 + \alpha j} \times \frac{(1 - \alpha j)}{(1 - \alpha j)}$$

$$z = \frac{j + \alpha - \alpha j^2 - \alpha^2 j}{1^2 + \alpha^2}$$

$$z = \frac{j(1 - \alpha^2) + 2\alpha}{1 + \alpha^2}$$

$$z = \frac{j(1 - e^{2j\theta}) + 2e^{j\theta}}{1 + e^{2j\theta}}$$

$$z = \frac{j(-2j\sin\theta\, e^{j\theta}) + 2e^{j\theta}}{2\cos\theta\, e^{j\theta}}$$

$$z = \frac{2e^{j\theta}(\sin\theta + 1)}{2\cos\theta\, e^{j\theta}}$$

$$z = \frac{1 + \sin\theta}{\cos\theta}$$

where $\theta = 2k\pi$ $(k = 0, 1, 2, \ldots n-1)$

10)

$$(z+j)^n + (z-j)^n = 0$$

$$(z+j)^n = -(z-j)^n$$

$$\left(\frac{z+j}{z-j}\right)^n = -1$$

$$\frac{z+j}{z-j} = \alpha$$

wher $\alpha = e^{j\theta}$

and $\theta = \frac{2\pi k}{n} + \frac{\pi}{n}$

$= \frac{(2k+1)\pi}{n}$

for $k = 0, 1, 2, \ldots\; n-1$

$$\frac{z+j}{z-j} = \alpha$$

$$z+j = \alpha(z-j)$$

$$z+j = \alpha z - \alpha j$$

$$\alpha j + j = \alpha z - z$$

$$j(1+\alpha) = z(\alpha - 1)$$

$$\frac{j(1+\alpha)}{\alpha - 1} = z$$

$$z = \frac{j \times 2\cos\frac{\theta}{2} e^{j\frac{\theta}{2}}}{-(1-\alpha)}$$

$$z = \frac{2j\cos\frac{\theta}{2} e^{j\frac{\theta}{2}}}{-(-2j\sin\frac{\theta}{2} e^{j\theta/2})}$$

$$z = \cot\frac{\theta}{2}$$

where $\theta = \frac{(2k+1)\pi}{n}$

for $k = 0, 1, 2, \ldots, n-1$